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REGULARITY OF THE OBSTACLE PROBLEM FOR A FRACTIONAL POWER OF THE LAPLACE OPERATOR

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Given a function φ and $s \in (0,1)$, we will study the solutions of the following obstacle problem

1. $u \geq \varphi$ in $\mathbb{R}^n$
2. $(-\triangle)^s u \geq 0$ in $\mathbb{R}^n$
3. $(-\triangle)^s u(x) = 0$ for those x such that $u(x) > \varphi(x)$
4. $\lim_{|x| \to +\infty} u(x) = 0$

We show that when φ is $C^{1,s}$ or smoother, the solution u is in the space $C^{1,\alpha}$ for every $\alpha < s$. In the case that the contact set $\{u = \varphi\}$ is convex, we prove the optimal regularity result $u \in C^{1,s}$. When φ is only $C^{1,\beta}$ for a $\beta < s$, we prove that our solution u is $C^{1,\alpha}$ for every $\alpha < \beta$.

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Chapter 1

Introduction

1.1 Set up of the problem

In this work, we will consider a function u that solves an obstacle problem for the operator $(-\triangle)^s$, for $s \in (0,1)$. Given a continuous function φ with a compact support (or at least rapid decay at infinity), we consider a continuous function u satisfying

$$u \geq \varphi \qquad \text{in } \mathbb{R}^n \tag{1.1.1}$$

$$(-\triangle)^s u \geq 0 \qquad \text{in } \mathbb{R}^n \tag{1.1.2}$$

$$(-\triangle)^s u(x) = 0 \qquad \text{for those } x \text{ such that } u(x) > \varphi(x) \tag{1.1.3}$$

$$\lim_{|x| \to +\infty} u(x) = 0 \tag{1.1.4}$$

When $\varphi \in C^\infty$, the expected optimal regularity for this type of problem is $C^{1,s}$. We prove $u \in C^{1,\alpha}$ for every $\alpha < s$. In the case when the contact set $\{u = \varphi\}$ is convex, we achieve the optimal result $u \in C^{1,s}$. If φ is only C^α for $\alpha < 1$ or Lipschitz, we will prove that u has the same modulus of continuity (Theorem 3.2.3). If φ is $C^{1,\beta}$, we will prove that $u \in C^{1,\alpha}$ for every $\alpha < \min(\beta, s)$ (Theorem 5.2.7).

The existence for such function u can be obtained by variational methods as the unique minimizer of

$$J(u) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \tag{1.1.5}$$

from all the functions u that satisfy $\varphi \leq u$ and are in a suitable function space.

We can also obtain u by a Perron's method approach, as the least supersolution of $(-\triangle)^s$ such that $u \geq \varphi$. Another approach is by choosing the optimal closed set $\Lambda \subset \mathbb{R}^n$ to maximize the solution of

1. $u(x) = \varphi(x)$ in Λ.
2. $(-\triangle)^s u = 0$ in $\mathbb{R}^n \setminus \Lambda$.
3. $\lim_{|x| \to \infty} u(x) = 0$.

We will choose the variational approach as the starting point. Then we will prove that u also solves the other two (equivalent) problem formulations. Our main focus, however, is the regularity of the solution.

Since we will be dealing with the operators $(-\triangle)^\sigma$, we will need several related results. Most of the present theory can be found in [9]. We will cite some results from there, and we will prove some others when we find it useful to present them in a form more convenient to our purposes. In chapter 2 we will study all the basic properties of these operators that we will need. In chapter 3, we will prove the existence of a solution u of our free boundary problem and we will prove the first regularity results. In chapter 4 we will obtain a better (nontrivial) regularity result, and at last, in chapter 5 we will present the optimal regularity result.

In the case $s = 1$, our problem turns into the usual obstacle problem. Given a domain $\Omega \subset \mathbb{R}^n$, and a function $\varphi : \Omega \to \mathbb{R}$, in the usual obstacle problem we have a function u which satisfies:

1. $u \geq \varphi$ in Ω.

2. $\triangle u \leq 0$ in Ω.

3. $\triangle u(x) = 0$ for those $x \in \Omega$ such that $u(x) > \varphi(x)$.

The existence of this problem can be obtained by minimizing a functional in H^1 with the constraint of $u \geq \varphi$ and some given boundary condition. If φ is a smooth function, then u is expected to be more regular than just in $H^1(\Omega)$. In 1971, Frehse [7] showed for the first time that u is as smooth as ϕ up to $C^{1,1}$, another proof was given in [5]. This regularity is optimal, simple examples show that for very smooth ϕ, u does not get any better than $C^{1,1}$.

Most of the regularity properties of the usual obstacle problem for the laplacian, including the regularity of the free boundary, can be found in [4].

Another related problem is the thin obstacle problem, or the Signorini problem. It is similar to the above problem, with the difference that the obstacle is now lower dimensional. In other words, the function φ is defined only in a hypersurface S of codimension 1, and we minimize the H^1 norm from all the functions u that are above the obstacle on S. Notice that u is well defined as a function in $H^{1/2}(S)$ due to the trace theorems. In a normalized case of this problem, the H^1 norm of u can be expressed in terms of the values that u attains on S. If we restrict out attention to S, we obtain a functional whose Euler Lagrange equation is an operator like $(-\triangle)^{1/2}$, that u will satisfy when it is above the obstacle φ. The optimal regularity for this problem is $C^{1,1/2}$, as it is shown in [12] for the two dimensional case, and very recently in [1] for the general case. We will continue with this problem in the next section.

1.2 Applications to the Signorini problem

Let us consider a smooth function with rapid decay at infinity $u_0 : \mathbb{R}^{n-1} \to \mathbb{R}$. Let $u : \mathbb{R}^{n-1} \times (0, \infty) \to \mathbb{R}$ be the unique solution of the laplace equation in the upper half space that vanishes at infinity with u_0 as the boundary condition:

$$
\begin{aligned}
u(x', 0) &= u_0(x') && \text{for } x' \in \mathbb{R}^{n-1} \\
\triangle u(x) &= 0 && \text{for } x \in \mathbb{R}^{n-1} \times (0, \infty)
\end{aligned}
$$

Consider the operator $T : u_0(x') \mapsto -\partial_n u(x', 0)$. We see that

$$
\int\limits_{\mathbb{R}^{n-1}} u(x', 0)(-\partial_n u(x', 0)) \, \mathrm{d}x' = \int\limits_{R^{n-1}\times(0,\infty)} -u(x)\triangle u(x) + |\nabla u(x)|^2 \, \mathrm{d}x
$$

$$
\int\limits_{R^{n-1}\times(0,\infty)} |\nabla u(x)|^2 \, \mathrm{d}x \geq 0
$$

Thus T is a positive operator. Moreover, since $\partial_n u(x)$ is also a harmonic function, if we apply the operator twice we get

$$
\begin{aligned}
T \circ T u_0 &= (-\partial_n)(-\partial_n) u(x', 0) = \partial_{nn} u(x', 0) \\
&= -\sum_{i=1}^{n-1} \partial_{ii} u(x', 0) \\
&= -\triangle u_0
\end{aligned}
$$

Therefore, the operator T that maps the Dirichlet type condition u_0 into the Neumann type $-\partial_n u$ is actually the operator $(-\triangle)^{1/2}$.

One version of the Signorini problem is this: given φ a smooth function in $\mathbb{R}^{n-1}$, the solution u of the Signorini problem is the least harmonic function in the upper half semispace $\mathbb{R}^{n-1} \times (0, \infty)$ such that $u \geq \varphi$ and $\partial_n u \leq 0$ on $\mathbb{R}^{n-1} \times \{0\}$. From the fact explained just above, we see that actually this problem is exactly our obstacle problem for the operator $(-\triangle)^{1/2}$. The regularity we obtained is therefore $C^{1,1/2}$ in the case $\{u = \varphi\}$ is convex, and $C^{1,\alpha}$ for every $\alpha < 1/2$ in the general case. The optimal regularity $C^{1,1/2}$ was obtained very recently for this problem by Athanasopoulos and Caffarelli in [1]. The two dimensional case, was proven previously by Richardson in [12]. In 1979, Caffarelli showed $C^{1,\alpha}$ regularity for a small value of α in the n dimensional case [3].

Usually the Signorini problem (or its equivalent formulation as the thin obstacle problem) is studied in bounded domains. For regularity purposes, one case can be deduced from the other. Suppose we have a solution of the

Signorini problem in a ball:

$$
\begin{aligned}
-\triangle u(x) &= 0 && \text{for } |x| < 1 \text{ and } x_n > 0 \\
u(x) &= 0 && \text{for } |x| = 1 \text{ and } x_n \geq 0 \\
u(x', 0) &\geq \varphi(x') && \text{for } |x'| \leq 1 \\
\partial_n u(x', 0) &\leq 0 && \text{for } |x'| \leq 1 \\
\partial_n u(x', 0) &= 0 && \text{where } u(x', 0) > \varphi
\end{aligned}
$$

For the problem to make sense, we assume that $\varphi(x') < 0$ when $|x'| = 1$. Let η be a radially symmetric cutoff function such that $\{\varphi > 0\} \subset\subset \{\eta = 1\}$ and $\operatorname{supp} \eta \subset B_1$. The function ηu is above φ and also satisfies $\partial_n \eta u(x', 0) \leq 0$ for $x' \in \mathbb{R}^{n-1}$ and $\partial_n \eta u(x', 0) = 0$ for those $x' \in \mathbb{R}^{n-1}$ such that $\eta u(x', 0) > \varphi(x')$. Although ηu may not be harmonic in the upper half space, its laplacian is a smooth function. Let v be the unique bounded solution of the Neumann type problem in the upper half semispace:

$$
\begin{aligned}
\triangle v(x) &= \triangle \eta u(x) = \triangle \eta(x)\, u(x) + 2 \nabla \eta(x) \cdot \nabla u(x) \\
\partial_n v(x', 0) &= 0
\end{aligned}
$$

Since $\triangle \eta u(x)$ is smooth and compactly supported, v is a smooth function. Now $\eta u - v$ is a solution of the Signorini problem without boundary with $\varphi - v$ as the obstacle. Therefore, we reduce the regularity for the bounded case, from the regularity of the unbounded case and our result applies.

1.3 Variations of the problem

There are small variations of the obstacle problem that can be considered. To simplify the variational proof of existence we could consider minimizers of the standard H^s norm from all the functions u that lie above a given obstacle φ. By the H^snorm we mean:

$$\|u\|_{H^s} = \sqrt{\int\limits_{\mathbb{R}^n} (1 + |\xi|^{2s}) \, |\hat{u}(\xi)|^s \, \mathrm{d}\xi}$$

In this case we obtain a free boundary problem of the sort

1. $u \geq \varphi$ in $\mathbb{R}^n$,
2. $u + (-\triangle)^s u \geq 0$ in $\mathbb{R}^n$,
3. $u + (-\triangle)^s u(x) = 0$ for those x such that $u(x) > \varphi(x)$.

The proofs of chapter 3 have to be adapted to use the operator $Id + (-\triangle)^s$ instead of $(-\triangle)^s$. Once we get that the solution u is semiconvex, then it is going to be Lipschitz and we can pass the term u to the right hand side, and everything in chapter 4 and 5 applies without changes. An advantage of this variation of the problem is that we can get existence also in the case $n = 1$ and $s > 1/2$.

We could also consider a problem with boundary values. Let φ be such that $\varphi(x) < 0$ for every $|x| \geq 1$. Let u be the minimizer of $J(u)$ (for J defined in (1.1.5)) from all the functions u that lie above φ and $u(x) = 0$ for every $x \in \mathbb{R}^n \setminus B_1$. Then we obtain the following free boundary problem

1. $u \geq \varphi$ in $\mathbb{R}^n$,

2. $u = 0$ in $\mathbb{R}^n \setminus B_1$,

3. $(-\triangle)^s u \geq 0$ in B_1,

4. $(-\triangle)^s u(x) = 0$ for those $x \in B_1$ such that $u(x) > \varphi(x)$.

With a trick like in 1.2, our result applies to the interior regularity of this problem. However, this solution u is not going to be $C^{1,\alpha}(\mathbb{R}^n)$ since it is not going to be differentiable across the boundary of the unit ball ∂B_1 (As a matter of fact, we cannot expect better that C^s on ∂B_1, the boundary regularity of the Dirichlet problem. See proposition 5.1.1).

1.4 Applications to mathematical finance

The operators $(-\triangle)^s$ arise in stochastic theory as the operators associated with symmetric α-stable Levy processes. Suppose we have such a Levy process X_t such that $X_0 = x$ for some point x in $\mathbb{R}^n$. We consider the optimal stopping time τ to maximize the function

$$u(x) = \sup_{\tau} E\left[\varphi(X_\tau)\ ;\ \tau < +\infty\right]$$

Then the function u turns out to be the solution of our obstacle problem

1. $u \geq \varphi$ in $\mathbb{R}^n$.

2. $(-\triangle)^s u \geq 0$ in $\mathbb{R}^n$.

3. $(-\triangle)^s u(x) = 0$ for those x such that $u(x) > \varphi(x)$.

4. $\lim_{|x|\to+\infty} u(x) = 0$

If on the other hand, we consider the following problem:

$$u(x) = \sup_{\tau} E\left[e^{-\lambda\tau}\varphi(X_\tau)\right]$$

Then the function u turns out to be the solution of the following obstacle problem

1. $u \geq \varphi$ in $\mathbb{R}^n$.

2. $\lambda u + (-\triangle)^s u \geq 0$ in $\mathbb{R}^n$.

3. $\lambda u + (-\triangle)^s u(x) = 0$ for those x such that $u(x) > \varphi(x)$.

4. $\lim_{|x|\to+\infty} u(x) = 0$

A problem like this arises in financial mathematics as a pricing model for American options. These models are of increasing interest in the last few years. The function u represents the rational price of a perpetual American option where the assets prices are modeled by a levy process X_t, and the payoff function is φ. For non perpetual options, a parabolic version of this problem is considered. A very readable explanation of these models can be found in the book of Cont and Tankov [6] (See also [10] and [11]). Usually the models are in one dimension, and although general payoffs functions are considered, the case when $\varphi = (K - e^x)^+$ (the American put) is of special interest.

There is not much work done regarding regularity. In [2], S. Boyarenko and S. Levendorskiĭ studied for what classes of Levy processes this problem has C^1 solutions (smooth pasting). They considered a very general family of (one dimensional) Levy processes, and a class of payoff functions that assures that the contact set is a half line.

When we consider jump processes whose corresponding integro differential operators have a kernel that coincides with $\frac{1}{|y|^{n+2s}}$ around the origin, then the solutions of the corresponding obstacle problem also satisfy an obstacle problem for the operator $(-\triangle)^s$ with a right hand side. In many cases, we can assure enough regularity for that right hand side and the results of this work hold for those intego-differential operators too.

Chapter 2

Preliminary properties of the fractional laplace operator

In this section, we provide some elementary properties of the operators $(-\triangle)^\sigma$ that we will need though this work. The usual reference for these operators is Landkof's book [9]. We will show how $(-\triangle)^\sigma$ interacts with C^α norms, and a characterization of its supersolutions.

2.1 Definitions and properties

Throughout this chapter $\mathcal{S}$ stands for the Schwartz space of rapidly decreasing C^∞ functions in $\mathbb{R}^n$. Its dual, written as $\mathcal{S}'$, is the space of tempered distributions in $\mathbb{R}^n$.

The following classical theorem about distributions is going to be used:

Theorem 2.1.1. *Suppose that a distribution f is such that for any nonnegative test function g, $\langle f, g\rangle \geq 0$. Then f is a nonnegative Radon measure in $\mathbb{R}^n$.*

Two distributions f and g in $\mathbb{R}^n$ are said to coincide in an open set Ω if for every test function ϕ supported inside Ω

$$\langle f, \phi\rangle = \langle g, \phi\rangle$$

We recall the definition of $(-\triangle)^\sigma$ as a pseudodifferential operator.

Definition 2.1.2. *Given $\sigma \in (-n/2, 1]$ and $f \in \mathcal{S}$, we define $(-\triangle)^\sigma f$ as:*

$$\widehat{(-\triangle)^\sigma} f(\xi) = |\xi|^{2\sigma} \hat{f}(\xi) \tag{2.1.1}$$

Notice that $(-\triangle)^\sigma f \notin \mathcal{S}$ since $|\xi|^{2\sigma}$ introduces a singularity at the origin in its Fourier transform. That singularity is going to translate in a lack of rapid decay for $(-\triangle)^\sigma f$. However, $(-\triangle)^\sigma f$ is still C^∞.

If $\sigma \leq -n/2$, then $|\xi|^{2\sigma}$ is not a tempered distribution, so we cannot allow that case. Technically, we could define the case $\sigma > 1$ this way, but we are not interested in this right now. Clearly, $(-\triangle)^1 = -\triangle$, $(-\triangle)^0 = Id$ and $(-\triangle)^{\sigma_1} \circ (-\triangle)^{\sigma_2} = (-\triangle)^{\sigma_1+\sigma_2}$.

We can also compute the same operator using a singular integral. When $f \in \mathcal{S}$ and $\sigma \in (0, 1)$, we can compute $(-\triangle)^\sigma f$ as:

$$(-\triangle)^\sigma f(x) = c_{n,\sigma} \mathrm{PV} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+2\sigma}} \, \mathrm{d}y \tag{2.1.2}$$

If $0 < \sigma < 1/2$, the singular integrals are clearly well defined for functions $f \in \mathcal{S}$. In case $1 > \sigma \geq 1/2$, there is a cancellation involved near $x = y$. In the case $\sigma < 1/2$, the integrand is in L^1, so the integral is not really "singular". The constant factor $c_{n,\sigma}$ degenerates when $\sigma \to 1$ or $\sigma \to 0$. Since linear functions vanish in (2.1.2), we may avoid using principal values by using the alternate form

$$(-\triangle)^\sigma f(x) = c_{n,\sigma} \int_{\mathbb{R}^n} \frac{f(x) - f(y) + \nabla f(x) \cdot (y-x)\chi_{|x-y|\leq 1}}{|x-y|^{n+2\sigma}} \, \mathrm{d}y$$

The operator $(-\triangle)^{-\sigma}$ (for $\sigma > 0$) can also be computed with an integral when $n > 2\sigma$ by

$$(-\triangle)^{-\sigma} f(x) = c_{n,-\sigma} \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2\sigma}} \, \mathrm{d}y \tag{2.1.3}$$

We refer to [9] for a detailed proof of the equivalence between (2.1.1) and (2.1.2) or (2.1.3).

From (2.1.3), we see that $F(x) = c_{n,-\sigma} \frac{1}{|x|^{n-2\sigma}}$ is the fundamental solution of $(-\triangle)^{\sigma}$, i.e. $(-\triangle)^{\sigma} F = \delta_0$ when $n > 2\sigma$. This function is generally known as the Riesz kernel.

From the formulas, we see the following trivial properties of $(-\triangle)^{\sigma}$.

1. $(-\triangle)^{\sigma}$ commutes with rigid motions.
2. $(-\triangle)^{\sigma}\big(u(\lambda .)\big)(x) = \lambda^{2\sigma}(-\triangle)^{\sigma} u(\lambda x)$
3. $\langle (-\triangle)^{\sigma} f, g \rangle = \langle f, (-\triangle)^{\sigma} g \rangle$, for any $f, g \in \mathcal{S}$.

From the definition of $(-\triangle)^{\sigma}$ in $\mathcal{S}$, we can extend it by duality in a large class of tempered distributions.

Definition 2.1.3. *Let $\bar{\mathcal{S}}_{\sigma}$ be the space of C^{∞} functions f such that $(1 + |x|^{n+2\sigma}) f^{(k)}(x)$ is bounded for every $k \geq 0$. We consider the topology in $\bar{\mathcal{S}}_{\sigma}$ given by the family of seminorms:*

$$[f]_k = \sup (1 + |x|^{n+2\sigma}) f^{(k)}(x)$$

And we take $\bar{\mathcal{S}}'_{\sigma}$ to be the dual of $\bar{\mathcal{S}}_{\sigma}$.

It is very simple to check that $(-\triangle)^\sigma f \in \bar{\mathcal{S}}_\sigma$ when $f \in \mathcal{S}$.

The symmetry of the operator $(-\triangle)^\sigma$ allows us to extend its definition to the space $\bar{\mathcal{S}}'_\sigma$ by duality. i.e. if $u \in \bar{\mathcal{S}}'_\sigma$

$$\langle (-\triangle)^\sigma u, f\rangle = \langle u, (-\triangle)^\sigma f\rangle$$

This definition coincides with the previous ones in the case $u \in \mathcal{S}$, and $(-\triangle)^\sigma$ is a continuous operator from $\bar{\mathcal{S}}'_\sigma$ to $\mathcal{S}'$.

We are rarely going to use these operator in such general spaces. But it is convenient to have in mind how far they can be extended. In general we will be applying these operators to functions in L^1_{loc}. The natural space that we are going to use is a weighted L^1 space:

$$L_\sigma := L^1_{loc} \cap \bar{\mathcal{S}}'_\sigma = \left\{ u : \mathbb{R}^n \to \mathbb{R} \text{ such that } \int\limits_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2\sigma}} \, \mathrm{d}x < +\infty \right\}$$

The norm in L_σ is naturally given by

$$\|u\|_{L_\sigma} = \int\limits_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2\sigma}} \, \mathrm{d}x$$

In special cases, our formulas with the Fourier transform or the singular integrals are enough to compute the value. The following technical property is intuitively obvious but it requires a proof because of the way we defined the operators.

Proposition 2.1.4. *Let f be a function in L_σ that is $C^{2\sigma+\varepsilon}$ (or $C^{1,2\sigma+\varepsilon-1}$ if $\sigma > 1/2$) for some $\varepsilon > 0$ in an open set Ω, then for $\sigma \in (0,1)$, $(-\triangle)^\sigma f$ is a continuous function in Ω and its values are given by the integral of* (2.1.2).

Proof. Let us take an arbitrary open set Ω_0 compactly contained in Ω. There exists a sequence $f_k \in S$ uniformly bounded in $C^{\sigma+\varepsilon}(\Omega)$ (or $C^{1,\sigma+\varepsilon-1}$), converging uniformly to f in Ω_0, and converging also to f in the norm of L_σ. By the uniform bound on the $C^{\sigma+\varepsilon}$ norm of f_k in Ω_0 we will shown that the integrals converge uniformly in Ω_0. Let, ε be any positive real number, then there is a $\rho > 0$ such that

$$\int\limits_{B_\rho} \frac{M}{|y|^{n-\varepsilon}}\,\mathrm{d}y \le \frac{\varepsilon}{3}$$

where $M = \sup[f_k]_{C^{\sigma+\varepsilon}}$. Now we split the integral for $(-\triangle)^\sigma f_k$ into the points close to x and the points far.

$$\begin{aligned}(-\triangle)^\sigma f_k(x) &= \int\limits_{\mathbb{R}^n} \frac{f_k(x)-f_k(y)}{|x-y|^{n+2\sigma}}\,\mathrm{d}y\\ &= \int\limits_{B_\rho(x)} \frac{f_k(x)-f_k(y)}{|x-y|^{n+2\sigma}}\,\mathrm{d}y + \int\limits_{\mathbb{R}^n\setminus B_\rho(x)} \frac{f_k(x)-f_k(y)}{|x-y|^{n+2\sigma}}\,\mathrm{d}y\\ &= I_1 + I_2\end{aligned}$$

For I_1, we can see it is small since

$$|I_1| \le \int\limits_{B_\rho} \frac{M}{|y|^{n-\varepsilon}}\,\mathrm{d}y \le \frac{\varepsilon}{3}$$

For I_2, the kernel $\frac{1}{|x-y|^{n+2\sigma}}$ is in $L^1(\mathbb{R}^n \setminus B_\rho(x))$, moreover since $f_k \to f$

in L_σ and $f_k(x) \to f(x)$, for k large enough

$$\left| I_2 - \int\limits_{\mathbb{R}^n \setminus B_\rho(x)} \frac{f(x)-f(y)}{|x-y|^{n+2\sigma}}\, \mathrm{d}y \right| \leq \left| \int\limits_{\mathbb{R}^n \setminus B_\rho(x)} \frac{f_k(x)-f(x)-f_k(y)+f(y)}{|x-y|^{n+2\sigma}}\, \mathrm{d}y \right| \leq \frac{\varepsilon}{3}$$

And we also know that

$$\left| \int\limits_{B_\rho(x)} \frac{f(x)-f(y)}{|x-y|^{n+2\sigma}}\, \mathrm{d}y \right| \leq \frac{\varepsilon}{3}$$

Then

$$\left| \int\limits_{\mathbb{R}^n} \frac{f_k(x)-f_k(y)}{|x-y|^{n+2\sigma}}\, \mathrm{d}y - \int\limits_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2\sigma}}\, \mathrm{d}y \right| \leq \varepsilon$$

Since ε can be chose arbitrarily small, the integrals converge

$$(-\triangle)^\sigma f_k(x) = \int\limits_{\mathbb{R}^n} \frac{f_k(x)-f_k(y)}{|x-y|^{n+2\sigma}}\, \mathrm{d}y \to \int\limits_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2\sigma}}\, \mathrm{d}y$$

But $(-\triangle)^\sigma f_k \to (-\triangle)^\sigma f$ in the topology of $\mathcal{S}'$. That implies that $(-\triangle)^\sigma f$ must coincide with the integral in Ω_0 by uniqueness of the limits. Besides, $(-\triangle)^\sigma f$ is continous in Ω_0 since it is the uniform limit of continuous functions.

Since Ω_0 is arbitrary, this happens for any $x \in \Omega$. $\square$

When we can use the singular integral representation of the operator $(-\triangle)^\sigma$, we obtain a very simple maximum principle.

Proposition 2.1.5. *Suppose that $u \in L_\sigma$, and there is a point x_0 such that:*

1. $u(x_0) = 0$

2. u is $C^{2\sigma+\varepsilon}$ (or $C^{1,2\sigma+\varepsilon-1}$ if $\sigma > 1/2$) for some $\varepsilon > 0$ in a neighborhood of x_0

3. $u \geq 0$ in $\mathbb{R}^n$

Then $(-\triangle)^\sigma u(x_0) \leq 0$. Moreover, $(-\triangle)^\sigma u(x_0) = 0$ only when $u \equiv 0$.

Proof. By proposition 2.1.4, $(-\triangle)^\sigma u$ is a continuous functions around x_0, and we can use the singular integral to compute its value at x_0.

$$\begin{aligned}(-\triangle)^\sigma u(x_0) &= \int_{\mathbb{R}^n} \frac{u(x_0) - u(y)}{|x_0 - y|^{n+2\sigma}} \,\mathrm{d}y \\ &\leq 0\end{aligned}$$

since we are integrating a nonpositive function. And the last inequality is clearly strict if $0 < u(y)$ in a set of positive measure. □

Corollary 2.1.6 (comparison principle for sufficiently smooth functions). *Suppose that $u, v \in L_\sigma$, and there is a point x_0 such that:*

1. $u(x_0) = v(x_0)$

2. u and v are $C^{2\sigma+\varepsilon}$ (or $C^{1,2\sigma+\varepsilon-1}$ if $\sigma \geq 1/2$) for some $\varepsilon > 0$ in a neighborhood of x_0

3. $u \geq v$ in $\mathbb{R}^n$

Then $(-\triangle)^\sigma u(x_0) \leq (-\triangle)^\sigma v(x_0)$. *Moreover,* $(-\triangle)^\sigma u(x_0) = (-\triangle)^\sigma v(x_0)$ *only when* u *and* v *coincide.*

The following propositions explain how the operators $(-\triangle)^\sigma$ interact with C^α norms:

Proposition 2.1.7. *Let* $u \in C^{0,\alpha}(\mathbb{R}^n)$*, for* $\alpha \in (0,1]$*, and* $\alpha > 2\sigma > 0$*, then* $(-\triangle)^\sigma u \in C^{0,\alpha-2\sigma}$ *and*

$$[(-\triangle)^\sigma u]_{C^{0,\alpha-2\sigma}} \leq C[u]_{C^{0,\alpha}}$$

where C *depends only on* α*,* σ *and* n*.*

Proof. For $x_1, x_2 \in \mathbb{R}^n$, let us estimate the difference $|(-\triangle)^\sigma u(x_1) - (-\triangle)^\sigma (x_2)|$,

$$\begin{aligned}
|(-\triangle)^\sigma u(x_1) - (-\triangle)^\sigma u(x_2)| &= C_{n,\sigma} \left| \int_{\mathbb{R}^n} \frac{u(x_1) - u(x_1+y) - u(x_2) + u(x_2+y)}{|y|^{n+2\sigma}} \, \mathrm{d}y \right| \\
&\leq I_1 + I_2
\end{aligned}$$

where

$$I_1 = C_{n,\sigma} \left| \int_{B_r} \frac{u(x_1) - u(x_1+y) - u(x_2) + u(x_2+y)}{|y|^{n+2\sigma}} \, \mathrm{d}y \right|$$

$$I_2 = C_{n,\sigma} \left| \int_{\mathbb{R}^n \setminus B_r} \frac{u(x_1) - u(x_1+y) - u(x_2) + u(x_2+y)}{|y|^{n+2\sigma}} \, \mathrm{d}y \right|$$

For I_1, we use that $|u(x_i) - u(x_i+y)| \leq [u]_{C^{0,\alpha}} |y|^\alpha$ for $i = 1, 2$. Therefore

$$\begin{aligned}
I_1 &\leq C_{n,\sigma} \left| \int_{B_r} \frac{2[u]_{C^{0,\alpha}} |y|^\alpha}{|y|^{n+2\sigma}} \right| \\
&\leq C[u]_{C^{0,\alpha}} r^{\alpha-2\sigma}
\end{aligned}$$

For I_2, we use that $|u(x_1+y)-u(x_2+y)| \leq [u]_{C^{0,\alpha}} |x_1-x_2|^\alpha$

$$\begin{aligned} I_1 &\leq C_{n,\sigma} \left| \int_{\mathbb{R}^n \setminus B_r} \frac{2[u]_{C^{0,\alpha}} |x_1-x_2|^\alpha}{|y|^{n+2\sigma}} \right| \\ &\leq C[u]_{C^{0,\alpha}} r^{-2\sigma} |x_1-x_2|^\alpha \end{aligned}$$

Picking $r = |x_1 - x_2|$, and adding I_1 with I_2 we obtain

$$|(-\triangle)^\sigma u(x_1) - (-\triangle)^\sigma(x_2)| \leq C[u]_{C^{0,\alpha}} |x_1-x_2|^{\alpha-2\sigma}$$

$\square$

Proposition 2.1.8. *Let $u \in C^{1,\alpha}(\mathbb{R}^n)$, for $\alpha \in (0,1]$, and $\sigma > 0$, then*

1. *If $\alpha > 2\sigma$, then $(-\triangle)^\sigma u \in C^{1,\alpha-2\sigma}$ and*

$$[(-\triangle)^\sigma u]_{C^{1,\alpha-2\sigma}} \leq C[u]_{C^{1,\alpha}}$$

where C depends only on α, σ and n.

2. *If $\alpha < 2\sigma$, then $(-\triangle)^\sigma u \in C^{0,\alpha-2\sigma+1}$ and*

$$[(-\triangle)^\sigma u]_{C^{0,\alpha-2\sigma+1}} \leq C[u]_{C^{1,\alpha}}$$

where C depends only on α, σ and n.

Proof. The first part follows simply by Proposition 2.1.7 plus the fact that the operators $(-\triangle)^\sigma$ commute with differentiation.

For the second part, let us first assume that $\sigma < 1/2$. We proceed like in the proof of Proposition 2.1.7, to get

$$|(-\triangle)^\sigma u(x_1) - (-\triangle)^\sigma(x_2)| \leq I_1 + I_2$$

for the same I_1 and I_2 as before. But now to estimate I_1 we use that since $u \in C^{1,\alpha}$,

$$\begin{aligned}|u(x_1) - u(x_1+y) - u(x_2) + u(x_2+y)| &\leq |(\nabla u(x_1) - \nabla u(x_2)) \cdot y| + [u]_{C^{1,\alpha}} |y|^{1+\alpha} \\ &\leq [u]_{C^{1,\alpha}} (|y|\,|x_1 - x_2|^{\alpha} + |y|^{1+\alpha})\end{aligned}$$

then $I_1 \leq C[u]_{C^{1,\alpha}}(r^{1-2\sigma} |x_1 - x_2|^{\alpha} + r^{1+\alpha-2\sigma})$.

In case $\sigma \geq 1/2$, we write $(-\triangle)^{\sigma} = (-\triangle)^{\sigma-1/2} \circ (-\triangle)^{1/2}$, and the result follows from the observation that $(-\triangle)^{1/2} = \sum_i R_i \partial_i$, where R_i are the Riesz transforms. $\square$

Iterating the last two Propositions we get the following result:

Proposition 2.1.9. *Let $u \in C^{k,\alpha}$, and suppose that $k + \alpha - 2\sigma$ is not an integer. Then $(-\triangle)^{\sigma} u \in C^{l,\beta}$ where l is the integer part of $k + \alpha - 2\sigma$ and $\beta = k + \alpha - 2\sigma - l$.*

Proposition 2.1.10. *Let $w = (-\triangle)^{\sigma} u$, Assume $w \in C^{0,\alpha}(\mathbb{R}^n)$ and $u \in L^{\infty}$, for $\alpha \in (0, 1]$, and $\sigma > 0$, then*

1. *If $\alpha + 2\sigma \leq 1$, then $u \in C^{0,\alpha+2\sigma}(\mathbb{R}^n)$. Moreover*

   $$\|u\|_{C^{0,\alpha+2\sigma}(\mathbb{R}^n)} \leq C(\|u\|_{L^{\infty}} + \|w\|_{C^{0,\alpha}})$$

   for a constant C depending only on n, α and σ.

2. *If $\alpha + 2\sigma > 1$, then $u \in C^{1,\alpha+2\sigma-1}(\mathbb{R}^n)$. Moreover*

   $$\|u\|_{C^{1,\alpha+2\sigma-1}(\mathbb{R}^n)} \leq C(\|u\|_{L^{\infty}} + \|w\|_{C^{0,\alpha}})$$

   for a constant C depending only on n, α and σ.

Proof. We will show that u has the corresponding regularity in a neighborhood of the origin. The same argument works for a neighborhood of every point, so we get respectively that $u \in C^{0,\alpha+2\sigma}(\mathbb{R}^n)$ or $u \in C^{1,\alpha+2\sigma-1}(\mathbb{R}^n)$.

Let η be a smooth cutoff function such that $\eta(x) \in [0,1]$ for every $x \in \mathbb{R}^n$, $\operatorname{supp} \eta \subset B_2$ and $\eta(x) = 1$ for every $x \in B_1$. Let

$$u_0(x) := c_{n,-\sigma} \int\limits_{\mathbb{R}^n} \frac{\eta(y) w(y)}{|x-y|^{n-2\sigma}} \, \mathrm{d}y = (-\triangle)^{-\sigma} \eta w(x)$$

Then $(-\triangle)^{\sigma} u_0 = w = (-\triangle)^{\sigma} u$ in B_1, and therefore $u - u_0$ is smooth in $B_{1/2}$. Moreover, its $C^{0,\alpha+2\sigma}$ or $C^{1,\alpha+2\sigma-1}$ norm can be estimated from the L^∞ norm of $u_0 - u$, that can be estimated from the L^∞ norms of u and w.

So, we are only left to show that $u_0 \in C^{0,\alpha+2\sigma}(B_{1/2})$. Assume $\alpha < 1$, then we write $u_0 = (\triangle)^{-\sigma} \eta w = (-\triangle)^{1-\sigma} \circ (-\triangle)^{-1} \eta w$, and from the $C^{2,\alpha}$ estimates for the Poisson equation (see [8]) we know that $(-\triangle)^{-1} \eta w \in C^{2,\alpha}$ and its norm depends only on $\|w\|_{C_{0,\alpha}}$. Now we apply Proposition 2.1.9 and we conclude the proof. On the other hand, if $\alpha = 1$, then $\alpha > 1 - 2\sigma$, and we write $u_0 = (-\triangle)^{-1} \circ (-\triangle)^{1-\sigma} \eta w$ and the result follows from Proposition 2.1.9 and the $C^{2,\alpha}$ estimates for the Poisson equation. □

Proposition 2.1.11. *Let $w = (-\triangle)^{\sigma} u$, Assume $w \in L^\infty(\mathbb{R}^n)$ and $u \in L^\infty$, for $\sigma > 0$, then*

1. *If $2\sigma \leq 1$, then $u \in C^{0,\alpha}(\mathbb{R}^n)$ for any $\alpha < 2\sigma$. Moreover*

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq C(\|u\|_{L^\infty} + \|w\|_{L^\infty})$$

for a constant C depending only on n, α and σ.

2. If $2\sigma > 1$, then $u \in C^{1,\alpha}(\mathbb{R}^n)$ for any $\alpha < 2\sigma - 1$. Moreover

$$\|u\|_{C^{1,\alpha}(\mathbb{R}^n)} \le C(\|u\|_{L^\infty} + \|w\|_{L^\infty})$$

for a constant C depending only on n, α and σ.

Proof. The proof is identical of the one of Proposition 2.1.10 with the difference that we have to use $C^{1,\alpha}$ estimates for the Poisson equation with L^∞ right hand side instead of $C^{2,\alpha}$ estimates. $\square$

Now we will explain the balayage problem for these operators.

Given a domain $\Omega \subset \mathbb{R}^n$, and a function $g \in \mathbb{R}^n - \Omega$ that is going to be regarded as the boundary condition, there is a unique solution of the following problem provided that g and Ω are regular enough:

$$\begin{aligned} u(x) &= g(x) && \text{when } x \in \mathbb{R}^n - \Omega \\ (-\triangle)^\sigma u(x) &= 0 && \text{when } x \in \Omega \end{aligned}$$

The solution of the balayage problem in a ball of radius r can be expressed explicitly using the following Poisson kernel:

$$P(x,y) = C_{n,\sigma} \left(\frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^\sigma \frac{1}{|x-y|^n} \text{ for } x \in B_r \text{ and } y \notin B_r \qquad (2.1.4)$$

where $C_{n,\sigma} = \Gamma\left(\frac{n}{2}\right) \pi^{-n/2-1} \sin(\pi\sigma)$.

Now, the solution of the balayage problem is given by:

$$u(x) = \int_{\mathbb{R}^n - B_r} P(x,y)\, g(y)\, \mathrm{d}y$$

The proof of this Poisson formula can be found in [9]. We will come back to the balayage problem in chapter 5.

2.2 Supersolutions and comparison

We want the least restrictive possible definition of supersolutions for the equation

$$(-\triangle)^\sigma u \geq 0 \tag{2.2.1}$$

so that we can prove general theorems of comparison. We want to be able to apply maximum principles to nonsmooth functions for which the integral representation (2.1.2) of $(-\triangle)^\sigma$ does not apply. We also want to be able to check (2.2.1) in an open domain Ω that is not the whole space $\mathbb{R}^n$. We will obtain characterizations of supersolutions similar to the mean value for superharmonic functions that we will use later in the paper.

When we are interested in the whole space, (2.2.1) means of course that $(-\triangle)^\sigma u$ is a nonnegative measure.

Definition 2.2.1. *We say that $u \in \bar{\mathcal{S}}'_\sigma$ satisfies $(-\triangle)^\sigma u \geq 0$ in an open set Ω if for every nonnegative test function ϕ whose support is inside Ω, $\langle u, (-\triangle)^\sigma \phi \rangle \geq 0$.*

The definition is saying that $(-\triangle)^\sigma u$ coincides with a nonnegative Radon measure in Ω. This is good for a definition but it is awkward to deal with. We would like to have a property like the definition of superharmonic functions comparing the value at a point with the means in small balls cen-

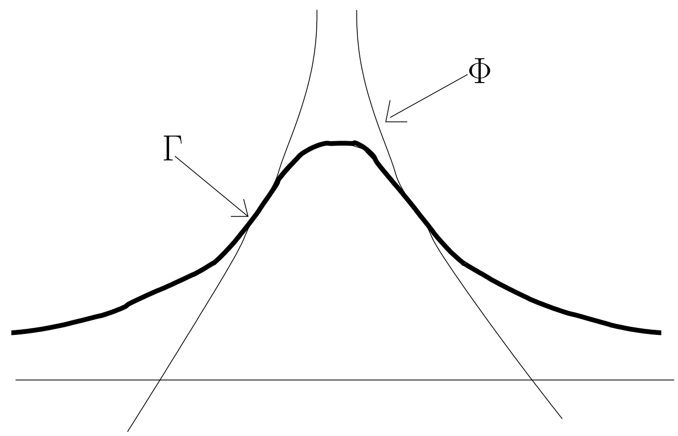

Figure 2.1: The function Γ.

tered there. We will restrict our study to functions $u \in L_\sigma$. We are going to use some special test functions.

Let $\Phi(x) = \frac{C}{|x|^{n-2\sigma}}$ be the fundamental solution of $(-\triangle)^\sigma$. Let us stick a paraboloid from below to cut out the singularity at $x = 0$ to obtain a $C^{1,1}$ function $\Gamma(x)$ that coincides with $\Phi(x)$ when x is outside the ball of radius one centered at the origin (see Figure 2.1).

Given $\lambda > 1$, consider $\Gamma_\lambda = \frac{1}{\lambda^{n-2\sigma}}\Gamma(\frac{x}{\lambda})$. The function $\Gamma_\lambda \in C^{1,1}$ coincides with Φ outside of the ball of radius λ centered at the origin, and it is a paraboloid inside that ball. Besides $\Gamma_{\lambda_1} \geq \Gamma_{\lambda_2}$ if $\lambda_1 \leq \lambda_2$.

We need the next proposition in order to use $(-\triangle)^\sigma\Gamma_\lambda$ as an approximation of the identity.

Proposition 2.2.2. $(-\triangle)^\sigma\Gamma$ *is a positive continuous function in* L^1. *And thus,* $(-\triangle)^\sigma\Gamma \geq 0$.

Besides $\int_{\mathbb{R}^n}(-\triangle)^\sigma\Gamma(x)\,\mathrm{d}x = 1$.

Proof. Since Γ is $C^{1,1}$, we can use the integral representation (2.1.2) to compute $(-\triangle)^\sigma\Gamma$.

If $x_0 \notin B_1$, then $\Gamma(x_0) = \Phi(x_0)$ and for every other x, $\Gamma(x) \le \Phi(x)$, then:

$$
\begin{aligned}
(-\triangle)^\sigma\Gamma(x_0) &= \int_{\mathbb{R}^n} \frac{\Gamma(x_0) - \Gamma(y)}{|x_0 - y|^{n+2\sigma}}\, \mathrm{d}y \\
&> \int_{\mathbb{R}^n} \frac{\Phi(x_0) - \Phi(y)}{|x_0 - y|^{n+2\sigma}}\, \mathrm{d}y = 0
\end{aligned}
$$

since Φ is the fundamental solution.

If $x_0 \in B_1 - \{0\}$, there exist and x_1 and a positive δ such that $\Phi(x - x_1) + \delta$ touches Γ from above at the point x_0. Now we use the singular integral representation:

$$
\begin{aligned}
(-\triangle)^\sigma\Gamma(x_0) &= \int_{\mathbb{R}^n} \frac{\Gamma(x_0) - \Gamma(y)}{|x_0 - y|^{n+2\sigma}}\, \mathrm{d}y \\
&> \int_{\mathbb{R}^n} \frac{\Phi(x_0 - x_1) + \delta - \Phi(y - x_1) - \delta}{|x_0 - y|^{n+2\sigma}}\, \mathrm{d}y = 0
\end{aligned}
$$

since $(-\triangle)^\sigma(\Phi(x_0 - x_1) + \delta) = 0$.

If $x_0 = 0$, then Γ attains its maximum at x_0:

$$
(-\triangle)^\sigma\Gamma(x_0) = \int_{\mathbb{R}^n} \frac{\Gamma(x_0) - \Gamma(y)}{|x_0 - y|^{n+2\sigma}}\, \mathrm{d}y > 0
$$

Because we are integrating a positive function.

To show that $\int_{\mathbb{R}^n} (-\triangle)^\sigma\Gamma(x)\, \mathrm{d}x = 1$ we consider a smooth cutoff function η such that $\eta(x) \le 1$ for every $x \in \mathbb{R}^n$, $\eta(x) = 1$ for every $x \in B_1$ and $\operatorname{supp}\eta \subset B_2$. Let $\eta_R(x) = \eta\left(\frac{x}{R}\right)$, then we have

$$\int_{\mathbb{R}^n} (-\triangle)^\sigma \Gamma(x)\,\mathrm{d}x - 1 = \lim_{R\to\infty} \langle (-\triangle)^\sigma \Gamma - (-\triangle)^\sigma \Phi, \eta_R \rangle =$$
$$= \lim_{R\to\infty} \langle \Gamma - \Phi, (-\triangle)^\sigma \eta_R \rangle$$
$$= 0$$

since clearly $(-\triangle)^\sigma \eta_R$ goes to zero uniformily on compact sets, and $\Gamma - \Phi$ is an L^1 function with compact support $\square$

Let $\gamma_\lambda = (-\triangle)^\sigma \Gamma_\lambda$.

Proposition 2.2.3. *For any λ, the function $\gamma_\lambda(x)$ decays like $\frac{1}{|x|^{n+2\sigma}}$ when $x \to \infty$.*

Proof. For $|x|$ large, $\Gamma_\lambda(x) = \Phi(x)$, and:

$$\begin{aligned}
\gamma_\lambda(x) &= \int \frac{\Gamma_\lambda(x) - \Gamma_\lambda(y)}{|x-y|^{n+2\sigma}}\,\mathrm{d}y \\
&= \int \frac{\Phi(x) - \Phi(y)}{|x-y|^{n+2\sigma}}\,\mathrm{d}y + \int \frac{\Phi(y) - \Gamma_\lambda(y)}{|x-y|^{n+2\sigma}}\,\mathrm{d}y \\
&= \int \frac{\Phi(y) - \Gamma_\lambda(y)}{|x-y|^{n+2\sigma}}\,\mathrm{d}y \\
&\cong \frac{1}{|x|^{n+2\sigma}}
\end{aligned}$$

since $\Phi(y) - \Gamma_\lambda(y)$ is a compactly supported function in L^1. $\square$

Proposition 2.2.4. *The family γ_λ is an approximation of the identity as $\lambda \to 0$. In the sense that*

$$u * \gamma_\lambda(x) = \int_{\mathbb{R}^n} u(y)\gamma_\lambda(x-y)\,\mathrm{d}y \to u(x) \text{ a.e. as } \lambda \to 0$$

Proof. First of all notice that $u(y)\gamma_\lambda(x-y)$ is integrable for every x since $u \in L_\sigma$ and γ_λ decays as $\frac{1}{1+|x|^{n+2\sigma}}$ by proposition 2.2.3.

We have to check the rescaling properties of $\gamma_\lambda = (-\triangle)^\sigma \Gamma_\lambda$.

$$\begin{aligned}\gamma_\lambda(x) = (-\triangle)^\sigma \Gamma_\lambda(x) &= (-\triangle)^\sigma \left(\frac{1}{\lambda^{n-2\sigma}} \Gamma\left(\frac{x}{\lambda}\right) \right) \\ &= \frac{1}{\lambda^n}\big((-\triangle)^\sigma \Gamma\big)\left(\frac{x}{\lambda}\right) = \frac{1}{\lambda^n}\gamma_1\left(\frac{x}{\lambda}\right)\end{aligned}$$

Since γ_1 is nonnegative and $\int \gamma_1 \, \mathrm{d}x = 1$, we conclude the proof. □

Proposition 2.2.5. *If $(-\triangle)^\sigma u$ is continuous at a point $x \in \mathbb{R}^n$ then*

$$(-\triangle)^\sigma u(x) = \lim_{\lambda \to 0} \frac{C}{\lambda^n}(u(x) - u * \gamma_\lambda(x)),$$

where the constant C depends only on σ and n.

Proof. Since $(-\triangle)^\sigma u$ is continuous at the point x, then it is bounded in a neighborhood of x and for any function $g \in L^1(\mathbb{R}^n)$ with compact support:

$$\lim_{\lambda \to 0} \int_{\mathbb{R}^n} (-\triangle)^\sigma u(x-y) \frac{1}{\lambda^n} g\left(\frac{y}{\lambda}\right) \, \mathrm{d}y = (-\triangle)^\sigma u(x) \cdot \int_{\mathbb{R}^n} g \, \mathrm{d}x \tag{2.2.2}$$

Let $g(y) = \Phi(y) - \Gamma(y)$, then

$$\begin{aligned}\frac{1}{\lambda^n} g\left(\frac{y}{\lambda}\right) &= \frac{1}{\lambda^n}\Phi\left(\frac{y}{\lambda}\right) - \frac{1}{\lambda^n}\Gamma\left(\frac{y}{\lambda}\right) \\ &= \frac{1}{\lambda^{2\sigma}}(\Phi(y) - \Gamma_\lambda(y))\end{aligned}$$

Thus, replacing in (2.2.2),

$$\begin{aligned}(-\triangle)^\sigma u(x) &= C \lim_{\lambda \to 0} \frac{1}{\lambda^{2\sigma}} \int_{\mathbb{R}^n} (-\triangle)^\sigma u(x-y)(\Phi(y) - \Gamma_\lambda(y)) \, \mathrm{d}y \\ &= \lim_{\lambda \to 0} \frac{C}{\lambda^{2\sigma}}(u(x) - u * \gamma_\lambda(x))\end{aligned}$$

$\square$

Proposition 2.2.6. *Given a function $u \in L_\sigma$, $(-\triangle)^\sigma u \geq 0$ in an open set Ω if and only if u is lower semicontinuous in Ω and*

$$u(x_0) \geq \int_{\mathbb{R}^n} u(x)\gamma_\lambda(x - x_0)\, \mathrm{d}x$$

for any x_0 in Ω and $\lambda \leq \mathrm{dist}(x_0, \partial\Omega)$.

Proof. We would like to test $(-\triangle)^\sigma u$ against $\Phi - \Gamma_\lambda$ and "integrate by parts". Unfortunately this may not be a valid test function. The next few paragraphs overcome this technical difficulty.

Let us consider a function u such that $\int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2\sigma}}\ \mathrm{d}x < +\infty$. If $r > \lambda_1 > \lambda_2$, $\Gamma_{\lambda_2} - \Gamma_{\lambda_1}$ is a nonnegative $C^{1,1}$ function supported in B_r. If $(-\triangle)^\sigma u \geq 0$ in $B_r(x_0)$ then:

$$\langle (-\triangle)^\sigma u, \Gamma_{\lambda_2}(x - x_0) - \Gamma_{\lambda_1}(x - x_0)\rangle \geq 0$$

Using the selfadjointness of $(-\triangle)^\sigma$:

$$\langle u, (-\triangle)^\sigma \Gamma_{\lambda_2}(x - x_0) - (-\triangle)^\sigma \Gamma_{\lambda_1}(x - x_0)\rangle \geq 0$$

Therefore

$$\langle u, \gamma_{\lambda_2}(x - x_0)\rangle \geq \langle u, \gamma_{\lambda_1}(x - x_0)\rangle$$
$$u * \gamma_{\lambda_2}(x_0) \geq u * \gamma_{\lambda_1}(x_0)$$

Let $\Omega_0 \subset\subset \Omega$, and $(-\triangle)^\sigma u \geq 0$ in Ω. Let $r = \mathrm{dist}(\Omega_0, \partial\Omega)$. Then if $r > \lambda_1 > \lambda_2 > 0$,

$$u * \gamma_{\lambda_1} \geq u * \gamma_{\lambda_2} \text{ in } \Omega_0 \tag{2.2.3}$$

But γ_λ is an approximate identity, $u * \gamma_\lambda \to u$ a.e. in Ω_0 as $\lambda \to 0$.

For each λ, $u * \gamma_\lambda$ is continuous. So u is the limit of an increasing sequence of continuous functions. That means that (possibly modifying u in a set of measure zero), u is lower semicontinuous.

Taking $\lambda_2 \to 0$ in (2.2.3), we obtain the important property of super-solutions of the operator $(-\triangle)^\sigma$ that replaces the mean value property of the classical Laplace operator:

$$u * \gamma_\lambda(x_0) \leq u(x_0) \text{ for every } x_0 \in \Omega \text{ and } \lambda \text{ small enough} \tag{2.2.4}$$

The *if* part is already proved (notice that γ is symmetric) when $\lambda < \operatorname{dist}(x_0, \partial\Omega)$. The case $\lambda = \operatorname{dist}(x_0, \partial\Omega)$ follows by passage to the limit. The *only if* part follows easily. □

Corollary 2.2.7. *There is a constant C such that for every $x \in \Omega$,*

$$u(x) \geq u * \gamma_\lambda(x) - C\lambda^{2s} \qquad \textit{for every } \lambda < \operatorname{dist}(x, \partial\Omega) \tag{2.2.5}$$

if and only if $(-\triangle)^\sigma u \geq -C$ in Ω (in the sense that $(-\triangle)^\sigma u + C$ is a non-negative Radon measure).

Proof. We can assume that Ω is bounded (since $f \geq -C$ locally in Ω is the same as $f \geq -C$ in the whole Ω for any distribution f).

Let $v = C\Phi * \chi_\Omega$, so that $(-\triangle)^\sigma v = C\chi_\Omega$.

By Proposition 2.2.5, for any $x \in \Omega$,

$$C = \lim_{\lambda \to 0} \frac{1}{\lambda^{2\sigma}} (v(x) - v * \gamma_\lambda(x))$$

But actually we can see that since $(-\triangle)^\sigma v$ is constant in Ω and $\Phi - \Gamma_\lambda$ is supported in B_λ, then

$$C = \frac{1}{\lambda^{2\sigma}}(v(x) - v * \gamma_\lambda(x)),$$

for $\lambda < \operatorname{dist}(x, \partial\Omega)$.

Now we consider $u + v$, then $u + v(x) \geq (u + v) * \gamma_\lambda(x)$ is equivalent to (2.2.5), that means that $(-\triangle)^\sigma(u + v) \geq 0$ in Ω. Thus (2.2.5) holds if and only if $(-\triangle)^\sigma(u + v) \geq 0$, i.e. $(-\triangle)^\sigma u \geq -C$ in Ω. $\square$

With Proposition 2.2.6 in mind, we can prove the basic properties of supersolutions for the operator $(-\triangle)^\sigma$ without requiring the singular integrals to be well defined. We now show a maximum principle.

Proposition 2.2.8. *Ley $\Omega \subset\subset \mathbb{R}^n$ be an open set, let u be a lower semicontinuous function in $\bar{\Omega}$ such that $(-\triangle)^\sigma u \geq 0$ in Ω and $u \geq 0$ in $\mathbb{R}^n \setminus \Omega$. Then $u \geq 0$ in $\mathbb{R}^n$. Moreover, if $u(x) = 0$ for one point x inside Ω, then $u \equiv 0$ in the whole $\mathbb{R}^n$.*

Proof. We need to require the semicontinuity in $\bar{\Omega}$ because we can not assure that any superharmonic function will be semicontinuous up to the boundary of the domain.

If u takes negative values in $\mathbb{R}^n$, then they must all lie inside Ω. Since u is lower semicontinous, it attains its minimum in $\bar{\Omega}$ (that is a compact set in $\mathbb{R}^n$). Suppose that the minimum is negative and is attained at a point $x_0 \in \Omega$.

Then by proposition 2.2.6, there is a λ such that:

$$u(x_0) \geq \int\limits_{\mathbb{R}^n} u(x)\gamma_\lambda(x - x_0)\, \mathrm{d}x$$

But γ_λ is strictly positive and has integral 1, then:

$$0 \geq \int\limits_{\mathbb{R}^n} (u(x) - u(x_0))\gamma_\lambda(x - x_0)\, \mathrm{d}x$$

That is impossible because since $u(x_0) < 0$, the right hand side is strictly positive.

Now, if $u(x_0) = 0$, we get

$$0 \geq \int\limits_{\mathbb{R}^n} u(x)\gamma_\lambda(x - x_0)\, \mathrm{d}x$$

But as $u(x)$ is nonnegative

$$0 \leq \int\limits_{\mathbb{R}^n} u(x)\gamma_\lambda(x - x_0)\, \mathrm{d}x$$

Therefore

$$0 = \int\limits_{\mathbb{R}^n} u(x)\gamma_\lambda(x - x_0)\, \mathrm{d}x$$

And since γ_λ is strictly positive we obtain $u \equiv 0$. $\square$

Proposition 2.2.9. *If $u_1, u_2 \in L_\sigma$ are two supersolution for the operator $(-\triangle)^\sigma$ in Ω (i.e. $(-\triangle)^\sigma u \geq 0$ and $(-\triangle)^\sigma v \geq 0$ in Ω), then so is $u(x) = \min(u_1(x), u_2(x))$.*

Proof. Given $x_0 \in \Omega$ then $u(x_0) = u_i(x_0)$ for $i = 1$ or $i = 2$. By proposition 2.2.6 for λ small enough

$$u_i(x_0) \geq \int_{\mathbb{R}^n} u_i(x)\gamma_\lambda(x - x_0)\,\mathrm{d}x$$

But $u_i(x_0) = u(x_0)$ and $u(x) \leq u_i(x)$ for every other x. Then:

$$u(x_0) \geq \int_{\mathbb{R}^n} u_i(x)\gamma_\lambda(x - x_0)\,\mathrm{d}x \geq \int_{\mathbb{R}^n} u(x)\gamma_\lambda(x - x_0)\,\mathrm{d}x$$

and $(-\triangle)^\sigma u \geq 0$. ∎

For functions u such that $(-\triangle)^\sigma u \leq 0$, a similar property holds:

Proposition 2.2.10. *Given function $u \in L_\sigma$, $(-\triangle)^\sigma u \leq 0$ in an open set Ω if and only if u is upper semicontinuous in Ω and $u * \gamma_\lambda(x_0) \geq u(x_0)$ for any x_0 in Ω and $\lambda \leq \operatorname{dist}(x_0, \partial\Omega)$.*

We can also obtain the analog of Corollary 2.2.7.

Corollary 2.2.11. *There is a constant C such that for every $x \in \Omega$ and $\lambda \leq \operatorname{dist}(x_0, \partial\Omega)$, $u(x) \leq u * \gamma_\lambda(x) + C\lambda^{2s}$, if and only if $(-\triangle)^\sigma u \leq C$ in Ω.*

Proposition 2.2.12. *Let $\Omega \subset\subset \mathbb{R}^n$ be an open set, $(-\triangle)^\sigma u \geq 0$ and $(-\triangle)^\sigma v \leq 0$ in Ω, such that $u \geq v$ in $\mathbb{R}^n \setminus \Omega$, and $u - v$ is lower semicontinuous in $\bar{\Omega}$. Then $u \geq v$ in $\mathbb{R}^n$. Moreover, if $u(x) = v(x)$ for one point x inside Ω, then $u \equiv v$ in the whole $\mathbb{R}^n$.*

Proof. Apply property 2.2.8 to $u - v$. ∎

We have a similar property for functions u such that $(-\triangle)^\sigma u = 0$ in an open set Ω.

Proposition 2.2.13. *Given function $u \in L_\sigma$, $(-\triangle)^\sigma u = 0$ in an open set Ω if and only if u is continuous in Ω and $u * \gamma_\lambda(x_0) = u(x_0)$ for any x_0 in Ω and $\lambda \leq \operatorname{dist}(x_0, \partial\Omega)$.*

From the above proposition, with a standard convolution argument, we can get an iterative gain in regularity and prove that a function u such that $(-\triangle)^\sigma u = 0$ in an open set Ω is C^∞ in that set. This is a well known result not only for the fractional laplacian, but for any pseudodifferential operator.

Remark 2.2.14. We are not going to compute γ_λ explicitly. The properties shown so far are enough for all our purposes. In [9], functions u such that $(-\triangle)^\sigma u \geq 0$ are defined in a similar (and equivalent) way using some function in place of γ_λ that is explicitly computed.

2.3 Stability properties

The purpose of this section is to obtain some cases in which we can pass to the limit the property of being a supersolution.

The main result is the following:

Proposition 2.3.1. *Let u_k be a bounded sequence in L_σ such that $(-\triangle)^\sigma u_k \geq 0$ in Ω for each k. Suppose that u_k converges to u in L^1_{loc} and that there is a function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\lim_{r\to\infty} \rho(r) = 0$ and*

$$\int_{\mathbb{R}^n \setminus B_r} \frac{|u_k(x)|}{1 + |x|^{n+2\sigma}} \, \mathrm{d}x \leq \rho(r)$$

for every k. Then $(-\triangle)^\sigma u \geq 0$ in Ω

Remark 2.3.2. The condition on the function ρ is to assure compactness in L_σ.

Before proving proposition 2.3.1 we will prove a stronger but less interesting result:

Proposition 2.3.3. *Let u_k be a sequence in $\bar{\mathcal{S}}'_\sigma$ such that $(-\triangle)^\sigma u_k \geq 0$ in Ω for each k. Suppose that u_k converges to u in $\bar{\mathcal{S}}'_\sigma$. Then $(-\triangle)^\sigma u \geq 0$ in Ω*

Proof. We have to check that for every nonnegative test function ϕ supported in Ω, $\langle(-\triangle)^\sigma u, \phi\rangle \geq 0$. But

$$\begin{aligned}\langle(-\triangle)^\sigma u, \phi\rangle = \langle u, (-\triangle)^\sigma \phi\rangle &= \lim_{k\to\infty} \langle u_k, (-\triangle)^\sigma \phi\rangle \\ &= \lim_{k\to\infty} \langle(-\triangle)^\sigma u_k, \phi\rangle \geq 0\end{aligned}$$

since $(-\triangle)^\sigma \phi \in \bar{S}$. ☐

The next lemma is what we need now to prove proposition 2.3.1

Lemma 2.3.4. *Let u_k be a bounded sequence in L_σ such that it converges in L^1_{loc} to a function u. Suppose that there is a function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\lim_{r\to\infty} \rho(r) = 0$ and*

$$\int\limits_{\mathbb{R}^n \setminus B_r} \frac{|u_k(x)|}{1 + |x|^{n+2\sigma}} \, \mathrm{d}x \leq \rho(r)$$

for every k. Then $u \in L_\sigma$ and u_k converges to u in $\bar{\mathcal{S}}'_\sigma$

Proof. The sequence u_k converges in L^1_{loc}, so we can extract a subsequence so that it converges almost everywhere. Then, applying Fatou's lemma and the uniform bound for $\|u_k\|_{L_\sigma}$:

$$\begin{aligned} C &\geq \liminf_{k\to\infty} \int\limits_{\mathbb{R}^n} \frac{|u_k(x)|}{1+|x|^{n+2\sigma}} \,\mathrm{d}x \\ &\geq \int\limits_{\mathbb{R}^n} \liminf_{k\to\infty} \frac{|u_k(x)|}{1+|x|^{n+2\sigma}} \,\mathrm{d}x \\ &= \int\limits_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2\sigma}} \,\mathrm{d}x \end{aligned}$$

So $u \in L_\sigma$ with the same bound on its norm. Similarly, it is shown that the inequality

$$\int\limits_{\mathbb{R}^n\setminus B_r} \frac{|u(x)|}{1+|x|^{n+2\sigma}} \,\mathrm{d}x \leq \rho(r)$$

holds also for u.

Let $\phi \in \mathcal{S}_\sigma$. We have to show that $\langle u_k, \phi\rangle \to \langle u, \phi\rangle$.

Since $\phi \in \mathcal{S}_\sigma$, then $\phi(x) \leq \frac{C}{1+|x|^{n+2\sigma}}$ for some $C > 0$. Consider a cutoff function θ such that:

1. $\theta \leq 1$ in $\mathbb{R}^n$,
2. $\theta = 1$ in B_1,
3. $\theta = 0$ in $\mathbb{R}^n \setminus B_2$.

Pick an arbitrary $r > 0$. Since $u \in L_\sigma$:

$$\int_{\mathbb{R}^n} u(x)\phi(x)\,\mathrm{d}x - \int_{\mathbb{R}^n} u_k(x)\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} (u(x) - u_k(x))\phi(x)\theta(x/r)\,\mathrm{d}x + \int_{\mathbb{R}^n} (u(x) - u_k(x))\phi(x)(1 - \theta(x/r))\,\mathrm{d}x$$

The first term goes to zero because $\phi(x)\theta(x/r)$ has compact support. The second term is less than $2C\rho(r)$, so it can be made arbitrarily small taking r large. ☐

Proof of proposition 2.3.1. Applying the lemma 2.3.4 we get that $u_k \to u$ in $\bar{\mathcal{S}}'_\sigma$, and then we can apply proposition 2.3.3 to get the result. ☐

Chapter 3

Basic properties of the free boundary problem

In this chapter we will construct a solution to our problem, and we will show the first regularity results.

3.1 Construction of the solution

We recall the statement of the problem that we are going to study.

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a continuous function with compact support [1], that we will consider the obstacle. We look for a continuous function u satisfying

1. $u \geq \varphi$ in $\mathbb{R}^n$,
2. $(-\triangle)^s u \geq 0$ in $\mathbb{R}^n$,
3. $(-\triangle)^s u(x) = 0$ for those x such that $u(x) > \varphi(x)$,
4. $\lim_{|x|\to+\infty} u(x) = 0$.

We will prove that for any such φ, there is a solution u to this problem. The proof fails when $n = 1$ and $s > 1/2$, because in that case it is impossible

[1] Rapid decay at infinity would suffice

to have $(-\triangle)^s u \geq 0$ in $\mathbb{R}^n$ and at the same time u to vanish at infinity.

We will construct u as the function that minimizes

$$J(u) := \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y \tag{3.1.1}$$

over all the functions $u \in \dot{H}^s$ that satisfy $\varphi \leq u$.

For any function $f \in \mathcal{S}$, the norm in $\dot{H}^s$ is given precisely by:

$$\|F\|_{\dot{H}^s} = \sqrt{\int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y} \tag{3.1.2}$$

That is equivalent to

$$\|f\|_{\dot{H}^s} \cong \sqrt{\int\limits_{\mathbb{R}^n} |\xi|^{2s} \left|\hat{f}(\xi)\right|^2 \, \mathrm{d}\xi} \tag{3.1.3}$$

In some texts, this space $\dot{H}^s$ is written as $\dot{L}^{2s,2}$.

When $n - 2s > 0$, the Sobolev embedding results say that $\dot{H}^s \subset L^{\frac{2n}{n-2s}}$ (see for example [14], chapter V.). The space $\dot{H}^s$ is defined as the completion of $\mathcal{S}$ with the norm $\|.\|_{\dot{H}^s}$. Indeed, $\dot{H}^s$ is the space of $L^{\frac{2n}{n-2s}}$ functions for which (3.1.2) is integrable. The space $\dot{H}^s$ is a Hilbert space with the inner product given by:

$$\begin{aligned}
\langle f, g \rangle_{\dot{H}^s} &= \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int\limits_{\mathbb{R}^n} f(x)(-\triangle)^s g(x) \, \mathrm{d}x \\
&\simeq \int\limits_{\mathbb{R}^n} |\xi|^{2s} \hat{f}(\xi)\overline{\hat{g}(\xi)} \, \mathrm{d}\xi
\end{aligned}$$

The set $\left\{u \in \dot{H}^s : \varphi \leq u\right\}$ is clearly convex and closed, and it is easy to show that it is nonempty because φ is bounded and has compact support. Thus, the (strictly convex) functional J has a unique minimum in that set. Let u be this minimizer. In the following propositions, we will prove that u is a solution of our obstacle problem.

Proposition 3.1.1. *The function u is a supersolution of $(-\triangle)^s u \geq 0$.*

Proof. Let h be any smooth nonnegative function with compact support, and $t > 0$. The function $u + th$ is above the obstacle, and so $\|u + th\|_{\dot{H}^s} \geq \|u\|_{\dot{H}^s}$. Therefore

$$\langle u, u \rangle_{\dot{H}^s} \leq \langle u + th, u + th \rangle_{\dot{H}^s}$$

$$\begin{aligned} 0 &\leq t\langle u, h\rangle_{\dot{H}^s} + t^2\langle h, h\rangle_{\dot{H}^s} \\ &\leq t\langle u, (-\triangle)^s h\rangle_{L^2} + t^2\langle h, h\rangle_{\dot{H}^s} \\ &\leq t\int u(-\triangle)^s h \, \mathrm{d}x + t^2\langle h, h\rangle_{\dot{H}^s} \end{aligned}$$

Letting $t \to 0^+$, we get that $\int u(-\triangle)^s h \, \mathrm{d}x = \int ((-\triangle)^s u) h \, \mathrm{d}x \geq 0$ for any nonnegative test function h. Therefore $(-\triangle)^s u$ is a nonnegative measure. $\square$

Corollary 3.1.2. *The function u is lower semicontinuous and the set $\{u > \varphi\}$ is open.*

Proof. Since $(-\triangle)^s u \geq 0$, by Proposition 2.2.6 u is lower semicontinuous. Thus $\{u > \varphi\}$ is open. $\square$

Proposition 3.1.3. *Let $x_0 \in \mathbb{R}^n$ such that $u(x_0) > \varphi(x_0)$. Let $r > 0$ such that $u > \varphi$ in $B_r(x_0)$, then $(-\triangle)^s u(x_0) = 0$ in $B_r(x_0)$.*

Proof. Since u is lower semicontinuous, the infimum of $u - \varphi$ is achieved in $B_r(x_0)$, then there is an $\varepsilon > 0$ such that $u > \varphi + \varepsilon$ in $B_r(x_0)$. For any continuous function h supported in $B_r(x_0)$, $u + th$ is going to be above φ for t small enough. So the same computation as in the proof of proposition 3.1.1 takes place. But this time t and h do not need to be nonnegative, and $(-\triangle)^s u = 0$ is obtained in $B_r(x_0)$. $\square$

The following proposition is a modification of a theorem of Evans for superharmonic functions. It will be used to prove continuity for u.

Proposition 3.1.4. *Let v be a bounded function in $\mathbb{R}^n$ such that $(-\triangle)^\sigma v \geq 0$ and $v_{|E}$ is continuous where $E = \operatorname{supp}(-\triangle)^\sigma u$. Then u is continuous in $\mathbb{R}^n$.*

Proof. In the open set $\mathbb{R}^n \setminus E$, $(-\triangle)^\sigma v = 0$, and thus v is continuous there. We are left to check that v is continuous in E.

Let $x_0 \in E$ and $x_k \to x_0$. Since v is lower semicontinuous $\liminf_{k\to\infty} v(x_k) \geq v(x_0)$. We are left to prove $\limsup_{k\to\infty} v(x_k) \leq v(x_0)$. Suppose the contrary, then we can extract a subsequence such that

$$\lim_{k\to\infty} v(x_k) = v(x_0) + a$$

where $a > 0$. Since v is continuous in E, then $x_k \notin E$ from some k on. We can assume that $x_k \notin E$ for any k by dropping the first few elements in the sequence.

Let y_k be the point in E closest to x_k (or one of them). Since v is continuous in E, $\lim v(y_k) = v(x_0)$.

Let $\lambda_k = |x_k - y_k| = \mathrm{dist}(x_k, E)$, so $\lambda_k \to 0$ as $k \to 0$.

Let $c_0 = \inf \frac{\gamma_1(x+e)}{\gamma_1(x)}$, where e is any unit vector. The infimum is achieved at one point, and thus $c_0 > 0$, since $\lim_{|x|\to\infty} \frac{\gamma_1(x+e)}{\gamma_1(x)} = 1$. By symmetry, the value of c_0 does not depend on e. Setting $e = \frac{x_k - y_k}{\lambda_k}$, we have

$$\begin{aligned}\gamma_{\lambda_k}(x - y_k) - c_0\gamma_{\lambda_k}(x - x_k) &= \frac{1}{\lambda_k^n}\left(\gamma_1\left(\frac{x - x_k}{\lambda_k} + \frac{x_k - y_k}{\lambda_k}\right) - c_0\gamma_1\left(\frac{x - x_k}{\lambda_k}\right)\right)\\ &\geq 0\end{aligned}$$

We now use Proposition 2.2.6 for $v(y_k)$ and Proposition 2.2.13 for $v(x_k)$ to get

$$\begin{aligned}v(y_k) &\geq \int\limits_{\mathbb{R}^n} \gamma_{\lambda_k}(x - y_k)v(x)\,\mathrm{d}x\\ &\geq \int\limits_{\mathbb{R}^n} c_0\gamma_{\lambda_k}(x - x_k)v(x)\,\mathrm{d}x + \int\limits_{\mathbb{R}^n}(\gamma_{\lambda_k}(x - y_k) - c_0\gamma_{\lambda_k}(x - x_k))v(x)\,\mathrm{d}x\\ &\geq c_0 v(x_k) + I_1 + I_2\end{aligned} \tag{3.1.4}$$

where

$$I_1 = \int\limits_{B_{\sqrt{\lambda_k}}(y_k)} (\gamma_{\lambda_k}(x - y_k) - c_0\gamma_{\lambda_k}(x - x_k))v(x)\,\mathrm{d}x \tag{3.1.5}$$

$$I_2 = \int\limits_{\mathbb{R}^n\setminus B_{\sqrt{\lambda_k}}(y_k)} (\gamma_{\lambda_k}(x - y_k) - c_0\gamma_{\lambda_k}(x - x_k))v(x)\,\mathrm{d}x \tag{3.1.6}$$

$$\tag{3.1.7}$$

Since v is lower semicontinuous, when $k \to \infty$, $\lambda_k \to 0$ and $v(x + y_k) \geq v(x_0) - \varepsilon_k$ for any $x \in B_{\sqrt{\lambda_k}(y_k)}$ and $\varepsilon_k \to 0$. Moreover, recalling that $\gamma_\lambda(x) = \frac{1}{\lambda^n}\gamma_1(x/\lambda)$ and $\int \gamma_1\,\mathrm{d}x = 1$, if we set $z = \frac{x-x_k}{\lambda_k}$ and $e = \frac{y_k-x_k}{\lambda_k}$,

$$\begin{aligned}\int\limits_{B_{\sqrt{\lambda}}(y_k)} (\gamma_{\lambda_k}(x - y_k) - c_0\gamma_{\lambda_k}(x - x_k))\,\mathrm{d}x &= \int\limits_{B_{\lambda^{-1/2}}} (\gamma_1(z) - c_0\gamma_1(z + e))\,\mathrm{d}z \\ &\geq 1 - c_0 - \tilde{\varepsilon}_k\end{aligned}$$

for $\tilde{\varepsilon}_k \to 0$ as $k \to \infty$. Combining these last two facts we obtain

$$I_1 \geq (1-c_0-\varepsilon_k)(v(x_0)-\varepsilon_k) = (1-c_0)v(x_0)-\tilde{\varepsilon}_k v(x_0)+\varepsilon_k(1-c_0-\tilde{\varepsilon}_k) \tag{3.1.8}$$

Now we estimate I_2,

$$I_2 \geq -\|v\|_{L^\infty} \int\limits_{\mathbb{R}^n\setminus B_{\sqrt{\lambda}}(y_k)} (\gamma_{\lambda_k}(x-y_k)-c_0\gamma_{\lambda_k}(x-x_k))\,\mathrm{d}x \geq -\tilde{\varepsilon}_k\|v\|_{L^\infty} \tag{3.1.9}$$

Substituting (3.1.8) and (3.1.9) in (3.1.4),

$$v(y_k) \geq c_0v(x_k) + (1 - c_0)v(x_0) - 2\tilde{\varepsilon}_k\|v\|_{L^\infty} + \varepsilon_k(1 - c_0 - \tilde{\varepsilon}_k)$$

where ε_k and $\tilde{\varepsilon}_k$ go to zero as $k \to \infty$. But this is a contradiction since $v(y_k) \to v(x_0)$ and $v(x_k) \to v(x_0) + a$ as $k \to \infty$. $\square$

Corollary 3.1.5. *The function u is continuous.*

In this way we finished proving that the minimizer of the functional $J(u)$ solves the original obstacle problem (1.1.1)-(1.1.4).

3.2 Semiconvexity

We are going to show that when φ is smooth enough, the solution u to our obstacle problem is Lipschitz and semiconvex. When φ has weaker smoothness assumptions, we will get correspondingly weaker conditions for u. The proofs in this section depend only on maximum principle and translation invariance. This regularity is common for all obstacle problems with operators satisfying these two conditions.

Proposition 3.2.1. *The function u is the least supersolution of $(-\triangle)^s u \geq 0$ that is above φ ($u \geq \varphi$) and is nonnegative at infinity ($\liminf_{|x|\to+\infty} u(x) \geq 0$).*

Proof. Let v be another supersolution of $v + (-\triangle)^s v \geq 0$ such that $v \geq \varphi$ and $\liminf_{|x|\to+\infty} v(x) \geq 0$. Let $m = \min(u, v)$. We want to show that actually $m = u$. By definition $m \leq u$, we are left to show $m \geq u$.

Since both u and v are supersolutions, by Proposition 2.2.9, so is m. The function m is also above φ because both u and v are, then m is another supersolution that is above φ. By Proposition 2.2.8, m is lower semicontinuous in $\mathbb{R}^n$.

Since $\varphi \leq m \leq u$, then $\lim_{x\to\infty} m(x) = 0$. For every x in the contact set $\{u = \varphi\}$, $m(x) = u(x)$. In $\Omega = \{u > \varphi\}$, u is a solution of $(-\triangle)^s u = 0$

and m a supersolution. By Corollary 3.1.5 u is continuous, then $m-u$ is lower semicontinuous. Then $m \geq u$ by comparison principle (Proposition 2.2.12) $\square$

Corollary 3.2.2. *The function u is bounded and $\sup u \leq \sup \varphi$.*

Proof. By hypothesis $u \geq 0$. The constant function $v(x) = \sup \varphi$ is a supersolution that is above φ. By Proposition 3.2.1, $u \leq v$ in $\mathbb{R}^n$. $\square$

Theorem 3.2.3. *If the obstacle φ has a modulus of continuity c, then the function u also has the same modulus of continuity.*

Proof. Since c is a modulus of continuity for φ, for any $h \in \mathbb{R}^n$, $\varphi(x+h) + c(|h|) \geq \varphi(x)$ for every $x \in \mathbb{R}^n$. Then the function $u(x+h) + c(|h|)$ is also a supersolution above φ. By Proposition 3.2.1, $u(x+h) + c(|h|) \geq u(x)$, for any $x, h \in \mathbb{R}^n$. Thus u has also c as a modulus of continuity. $\square$

Corollary 3.2.4. *The function u is Lipschitz, and its Lipschitz constant is not larger than the one of φ.*

Proof. It follows from Proposition 3.2.3 with $c(r) = Cr$. $\square$

Proposition 3.2.5. *Suppose that $\varphi \in C^{1,1}$. For any vector $e \in \mathbb{R}^n$, let $C = \sup -\partial_{ee}\varphi$. Then $\partial_{ee} u \geq -C$ too. Thus, u is semiconvex, and therefore for any point $x \in \mathbb{R}^n$, there is a paraboloid of opening C touching u from below.*

Proof. Since $\partial_{ee}\varphi \geq -C$, then we look at the second incremental quotients:

$$\frac{\varphi(x+te) + \varphi(x-te)}{2} + Ct^2 \geq \varphi$$

for every $t > 0$ and $x \in \mathbb{R}^n$. Therefore

$$v(x) := \frac{u(x+te) + u(x-te)}{2} + Ct^2 \geq \varphi$$

and v is also a supersolution of $(-\triangle)^s v \geq 0$. By Proposition 3.2.1, $v \geq u$, then

$$v(x) = \frac{u(x+te) + u(x-te)}{2} + Ct^2 \geq u(x)$$

for every v $t > 0$ and $x \in \mathbb{R}^n$. Therefore $\partial_{ee} u \geq -C$. $\square$

The above proposition is enough to treat the case when φ is $C^{1,1}$. However, to obtain the sharp estimates for $\varphi \in C^{1,\alpha}$, we need to refine the previous result. The following propositions are proven more or less with the same idea as in Proposition 3.2.5, but with a different thing instead of the second incremental quotient.

Proposition 3.2.6. *Let us suppose that $(-\triangle)^\sigma \varphi \leq C$ for some constant C and some $\sigma \in (0,1)$. Then also $(-\triangle)^\sigma u \leq C$ (maybe for another C depending on the dimension n).*

Proof. We apply Corollary 2.2.11 instead of the second order incremental quotient to obtain:

$$\varphi(x) \leq \varphi * \gamma_\lambda(x) + C\lambda^{2\sigma}$$

for any $x \in \mathbb{R}^n$ and any λ. Since $(-\triangle)^s(u * \gamma_\lambda + C\lambda^{2\sigma}) = (-\triangle)^s u * \gamma_\lambda \geq 0$, and

$$u * \gamma_\lambda + C\lambda^{2\sigma} \geq \varphi * \gamma_\lambda + C\lambda^{2\sigma} \geq \varphi(x),$$

then $u * \gamma_\lambda + C\lambda^{2\sigma} \geq u$ by Proposition 3.2.1. Thus $(-\triangle)^s u \geq -C$ by Corollary 2.2.7. $\square$

Proposition 3.2.7. *If $(-\triangle)^s\varphi \in L^\infty(\mathbb{R}^n)$, then $(-\triangle)^s u \in L^\infty(\mathbb{R}^n)$.*

Proof. This proposition follows by combining Proposition 3.2.6 with the fact that $(-\triangle)^s u \geq 0$. $\square$

Proposition 3.2.8. *If $\varphi \in C^{1,\alpha}$, then for every $x_0 \in \mathbb{R}^n$, there is a vector $a \in \mathbb{R}^n$ such that*

$$u(x_0 + h) \geq u(x_0) + a \cdot h - C\,|h|^{1+\alpha} \quad \text{for every } h \in \mathbb{R}^n$$

i.e. the function u has a supporting plane at each point with an error of order $1+\alpha$.

To prove this proposition we will need a couple of lemmas.

Lemma 3.2.9. *Suppose $\varphi \in C^{1+\alpha}$. If $0 = \sum \lambda_j h_j$ for $h_j \in \mathbb{R}^n$ and $\lambda_j \in [0,1]$ such that $\sum \lambda_j = 1$, then*

$$u(x) \leq \sum \lambda_j u(x + h_j) + C \sum \lambda_j\,|h_j|^{1+\alpha} \quad \text{for any } x \in \mathbb{R}^n \tag{3.2.1}$$

Proof. Since $\varphi \in C^{1+\alpha}$, there is a constant C for which

$$\begin{aligned}
&\sum \lambda_j \varphi(x + h_j) + C \sum \lambda_j\,|h_j|^{1+\alpha} \\
&\qquad \geq \sum \lambda_j (\varphi(x) + \nabla\varphi(x) \cdot h_j - C\,|h_j|^{1+\alpha}) + C \sum \lambda_j\,|h_j|^{1+\alpha} \\
&\qquad \geq \varphi(x) + \nabla\varphi(x) \cdot \sum \lambda_j h_j = \varphi(x)
\end{aligned}$$

Like in the proof of proposition 3.2.5, we obtain that

$$v(x) = \sum \lambda_j u(x + h_j) + C \sum \lambda_j\,|h_j|^{1+\alpha}$$

is a function above φ such that $(-\triangle)^s v \geq 0$, and thus it is also above u. By Proposition 3.2.1,

$$\sum \lambda_j u(x+h_j) + C \sum \lambda_j |h_j|^{1+\alpha} \geq u(x),$$

as we wanted to show. $\square$

Lemma 3.2.10. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function that satisfies an inequality like* (3.2.1)*. Then for every $x \in \mathbb{R}$, f has right and left derivatives at x. Moreover, the right derivative is greater than the left derivative, and for any number a in the closed interval between them and for every $h \in \mathbb{R}$:*

$$f(x+h) \geq f(x) + a \cdot h - C|h|^{1+\alpha} \tag{3.2.2}$$

where C depends only on the constant of the inequality (3.2.1)*.*

Proof. We will show that f has derivatives from both sides, and the one from the left is smaller or equal than the one from the right. Then we will show that (3.2.2) holds for any a between the two derivatives.

Let x_0 be any point in $\mathbb{R}$, and $0 < h' < h$. Consider the inequality (3.2.1) with $x = x_0 + h'$, $h_1 = -h'$, $h_2 = h - h'$, and thus $\lambda_1 = \frac{h-h'}{h}$ and $\lambda_2 = \frac{h'}{h}$. We have

$$\begin{aligned} f(x_0+h') &\leq \frac{h-h'}{h} f(x_0) + \frac{h'}{h} f(x+h) + C\left(\frac{h-h'}{h}|h'|^{1+\alpha} + \frac{h'}{h}|h-h'|^{1+\alpha}\right) \\ &\leq f(x) + \frac{h'}{h}(f(x+h) - f(h)) + Ch' \cdot h^\alpha \end{aligned}$$

Then,

$$\frac{f(x+h') - f(x)}{h'} \leq \frac{f(x+h) - f(x)}{h} + Ch^\alpha \tag{3.2.3}$$

Since f is Lipschitz, its incremental quotients are bounded, and then (3.2.3) implies that $\frac{f(x+h)-f(x)}{h}$ has a limit as $h \to 0^+$. Thus the right derivative exists.

Similarly, we can show that the left derivative exists. We want to see now that the right derivative is greater or equal than the left one.

Consider $h_1 = h$ and $h_2 = -h$ in inequality (3.2.1), then

$$f(x) \leq \frac{1}{2} f(x+h) + \frac{1}{2} f(x-h) + Ch^{1+\alpha}$$

Therefore,

$$\frac{f(x) - f(x-h)}{h} \leq \frac{f(x+h) - f(x)}{h} + 2Ch^{\alpha}$$

Taking $h \to 0$, we obtain that the right derivative is not less than the left one. Let a be a real number that is not greater than the right derivative, and no less than the left derivative. Taking $h' \to 0$ in (3.2.3),

$$a \leq \frac{f(x+h) - f(x)}{h} + Ch^{\alpha}$$

Then

$$f(x+h) \geq f(x) + a \cdot h - Ch^{1+\alpha}$$

The result follows similarly for negative h. □

We are now ready to prove Proposition 3.2.8.

Proof of Proposition 3.2.8. By Lemma 3.2.9 we have that if $0 = \sum \lambda_j h_j$ for $h_j \in \mathbb{R}^n$ and $\lambda_j \in [0,1]$ such that $\sum \lambda_j = 1$, then

$$u(x) \leq \sum \lambda_j u(x+h_j) + C \sum \lambda_j |h_j|^{1+\alpha} \quad \text{for any } x \in \mathbb{R}^n \tag{3.2.4}$$

Let us assume, without loss of generality, that $x_0 = 0$. For any unit vector e, we apply Lemma 3.2.10 to see that u satisfies the inequality (3.2.1) in every ray from the origin with possibly a different linear part. That means that for each $e \in S^{n-1}$, there is a real number $a(e)$ such that

$$u(te) \geq u(0) + a(e) \cdot t - C\,|t|^{1+\alpha} \tag{3.2.5}$$

where the constant C does not depend on e and $a(e)$ is the directional derivative

$$a(e) = \lim_{t \to 0^+} \frac{u(te) - u(0)}{t} \tag{3.2.6}$$

Now we will show that the homogeneous of degree one function $c(x) = a\left(\frac{x}{|x|}\right) \cdot |x|$ is convex. We have to see that $c(\lambda x + (1-\lambda)y) \leq \lambda c(x) + (1-\lambda)c(y)$. We use the inequality (3.2.4) to obtain:

$$u(\lambda tx + (1-\lambda)ty) \leq \lambda u(tx) + (1-\lambda)u(ty) + C(t\,|x-y|)^{1+\alpha},$$

for any real number t. We now subtract $u(0)$ on both sides and divide by t to obtain:

$$\frac{u(\lambda tx + (1-\lambda)ty) - u(0)}{t} \leq \lambda \frac{u(tx) - u(0)}{t} + (1-\lambda)\frac{u(ty) - u(0)}{t} + Ct^{\alpha}(|x-y|)^{1+\alpha}$$

Now we take limit as $t \to 0$, and replace by the corresponding value of c using the directional derivatives of u to obtain:

$$c(\lambda x + (1-\lambda)y) \leq \lambda c(x) + (1-\lambda)c(y)$$

Now that we know that c is convex, let a be the slope of a supporting plane at the origin (a is a vector in the subdifferential of c at zero). then

$c(h) \geq a \cdot h$ for every $h \in \mathbb{R}^n$. Therefore, recalling (3.2.5),

$$\begin{aligned} u(h) &\geq u(0) + c(h) - C\,|h|^{1+\alpha} \\ &\geq u(0) + a \cdot h - C\,|h|^{1+\alpha} \end{aligned}$$

which concludes the proof. □

We finish this section by showing that u solves the third specification of the problem

Proposition 3.2.11. *For any closed set $\Lambda \subset \mathbb{R}^n$, let v be the solution of*

1. $v(x) = \varphi(x)$ *in* Λ.
2. $(-\triangle)^s v = 0$ *in* $\mathbb{R}^n \setminus \Lambda$.
3. $\lim_{|x|\to\infty} v(x) = 0$

then $v \leq u$ in $\mathbb{R}^n$ (obviously, in the case $\Lambda = \{u = \varphi\}$, $u = v$).

Proof. Since $u \geq \varphi$ in $\mathbb{R}^n$, then $u \geq v$ in Λ. Since u is a supersolution of $(-\triangle)^s u \geq 0$ in $\mathbb{R}^n \setminus \Lambda$ and v is a solution, then $u \geq v$ in $\mathbb{R}^n \setminus \Lambda$ by maximum principle. □

Chapter 4

An improvement in regularity

In the rest of this work, we will study further regularity results for u. This chapter is devoted to show that $(-\triangle)^s u$ is C^α for some small α if the obstacle φ is smooth enough.

4.1 Problem

To study the regularity of the problem, it is convenient to consider the difference $u - \varphi$. The problem that we obtain is also an obstacle problem, the obstacle is zero, but there is a right hand side for the equation. For convenience, we will continue to call u to our solution, although the problem is slightly different.

Thus, we have u to be a solution of the following free boundary problem:

$$u(x) \geq 0 \tag{4.1.1}$$

$$(-\triangle)^s u(x) \geq \phi(x) \tag{4.1.2}$$

$$(-\triangle)^s u(x) = \phi(x) \text{ when } u > 0 \tag{4.1.3}$$

where ϕ is $-(-\triangle)^s \varphi$, for the obstacle φ of the previous chapters.

Since we are going to be using $(-\triangle)^s u(x)$ a great deal, we define $w = (-\triangle)^s u$.

If we assume φ to be C^∞, the right hand side ϕ will be C^∞ as well. However, it will be enough for all the proofs to come (and actually more than enough) to assume ϕ to be just Lipschitz.

If we assume $\varphi \in C^{1,\beta}$ for some $\beta \in (0,1)$ such that $1+\beta > 2s$, we will have $\phi \in C^{1+\beta-2s}$. The results of this chapter will still apply for this case, but the proofs have some extra complications. We will address this case at the end of the chapter. At first we will assume φ to be C^∞, and thus ϕ to be Lipschitz.

In this situation, we know from chapter 3 that the function u is a bounded Lipschitz function (Proposition 3.2.4) and also semiconvex (Proposition 3.2.5), and so far we know that w is L^∞ from corollary 3.2.7 . We want to prove that w is Hölder continuous. To summarize, we know:

$$\sup_x |\phi(x)| \le C \tag{4.1.4}$$

$$\sup_{x,y} \frac{|\phi(x)-\phi(y)|}{|x-y|} \le C \tag{4.1.5}$$

$$u_{ee} \ge -C \text{ for every } e \text{ such that } |e| = 1 \tag{4.1.6}$$

$$|w(x)| \le C \tag{4.1.7}$$

for some constant C.

4.2 A few lemmas

Lemma 4.2.1. *For $\sigma \in (0,1)$, there is a constant C depending only on σ and dimension, such that if v is bounded and semiconvex,*

$$\sup_x |v(x)| \leq A \tag{4.2.1}$$

$$\inf_x \inf_{|e|=1} v_{ee}(x) \geq -B \tag{4.2.2}$$

then $\sup_x (-\triangle)^\sigma v(x) \leq C \cdot B^\sigma \cdot A^{1-\sigma}$.

Proof. We can assume that v is smooth. Otherwise we take a smooth compactly supported function ψ with integral one, and consider the approximation of the identity $\psi_\lambda(x) = \lambda^n \psi(\lambda x)$. Then for any $\lambda > 0$, $\psi_\lambda * v$ satisfies (4.2.1) and (4.2.2) and it is a smooth function. If we can obtain that $\sup_x |(-\triangle)^\sigma (\psi_\lambda * v)(x)| \leq C \cdot B^\sigma \cdot A^{1-\sigma}$ uniformly in λ, then we pass to the limit as $\lambda \to 0$. Therefore, it is enough to prove the lemma for smooth v.

The value of $(-\triangle)^\sigma v(x)$ can be computed with the integral

$$\begin{aligned}
(-\triangle)^\sigma v(x) &= \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x-y|^{n+2\sigma}} \,\mathrm{d}y \\
&\leq \int_{B_R(x)} \frac{v(x) - v(x) - \nabla v(x) \cdot (y-x) + B\,|y-x|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n \setminus B_R(x)} \frac{A}{|x-y|^{n+2\sigma}} \,\mathrm{d}y \\
&\leq B \left(\int_{B_R(x)} \frac{1}{|x-y|^{n+2\sigma-2}} \,\mathrm{d}y \right) + A \left(\int_{\mathbb{R}^n \setminus B_R(x)} \frac{1}{|x-y|^{n+2\sigma}} \,\mathrm{d}y \right) \\
&\leq C \left(B \cdot R^{2-2\sigma} + A \cdot R^{-2\sigma} \right)
\end{aligned}$$

Replacing R by $\sqrt{\frac{A}{B}}$ in the above inequality we obtain:

$$(-\triangle)^\sigma v(x) \le C B^\sigma A^{1-\sigma}$$

□

Remark 4.2.2. Actually in lemma 4.2.1, the condition (4.2.2) could be replaced by $\triangle v \ge -B$ by using the fact that

$$\triangle v = 2n \lim_{r \to 0^+} r^{-2} \left(\frac{1}{|\partial B_r|} \int_{\partial B_r} v(y) - v(x) \, \mathrm{d}y \right)$$

Lemma 4.2.3. *Let v be such that $\|(-\triangle)^\sigma v(x)\|_\infty \le C$. For $\alpha < 2\sigma$, consider $v_\lambda = \frac{1}{\lambda^\alpha} v(\lambda x)$. Then $\|(-\triangle)^\sigma v_\lambda(x)\|_\infty \to 0$ as $\lambda \to 0$.*

Proof. Recalling that $(-\triangle)^\sigma (v(\lambda x)) = \lambda^{2r} \left((-\triangle)^r v\right)(\lambda x)$,

$$\begin{aligned} |(-\triangle)^\sigma v_\lambda(x)| &= \left|\lambda^{-\alpha} \lambda^{2r} \left((-\triangle)^{2r} v\right)(\lambda x)\right| \\ &\le C\lambda^{2r-\alpha} \to 0 \end{aligned}$$

□

Lemma 4.2.4. *For any $\sigma \in (0,1)$ and $\delta > 0$, if ε and α are chosen small enough then there is a $\gamma > 0$ such that if*

$$(-\triangle)^\sigma v(x) \le \varepsilon \text{ for } x \in B_1 \tag{4.2.3}$$

$$v(x) \le 1 \text{ for } x \in B_1 \tag{4.2.4}$$

$$v(x) \le |2x|^\alpha \text{ for } x \in \mathbb{R}^n \setminus B_1 \tag{4.2.5}$$

$$\delta \le |\{x \in B_1 : v(x) \le 0\}| \tag{4.2.6}$$

then $v(x) \le 1 - \gamma$ for $x \in B_{1/2}$.

Proof. As in lemma 4.2.1, we can assume v to be smooth.

Let $b(x) = \beta(|x|)$ be a fixed smooth radial function with support in B_1 such that $\beta(0) = 1$ and β is monotone decreasing.

For small enough ε and α, we can choose a positive number $\kappa < 1/2$, such that

$$\varepsilon + \kappa \sup_x (-\triangle)^\sigma b(x) + \int_{\mathbb{R}^n - B_{\frac{1}{4}}} \frac{|4y|^\alpha - 1}{|y|^{n+2\sigma}} \, \mathrm{d}y < \frac{\delta}{2 \cdot 2^{n+2\sigma}} \tag{4.2.7}$$

Let $\gamma = \kappa(\beta(1/2) - \beta(3/4))$. Suppose there is a point $x_0 \in B_{1/2}$ such that $v(x_0) > 1 - \gamma = 1 - \kappa \ \beta(1/2) + \kappa \ \beta(3/4)$. Then $v(x_0) + \kappa b(x_0) \geq 1 + \kappa \ \beta(3/4)$, that is larger than $v(y) + \kappa b(y)$ for any $y \in B_1 \setminus B_{3/4}$. This means that the supremum of $v(x) + \kappa b(x)$ for $x \in B_1$ is greater than 1 and is achieved in an interior point of $B_{3/4}$. Let us call that point x_1. Now we will evaluate $(-\triangle)^\sigma (v + \kappa b)(x_1)$.

On one hand, $(-\triangle)^\sigma (v + \kappa b)(x_1) = (-\triangle)^\sigma v(x_1) + \kappa(-\triangle)^\sigma b(x_1) \leq \varepsilon + \kappa(-\triangle)^\sigma b(x_1)$.

On the other hand we have

$$(-\triangle)^\sigma (v + \kappa b)(x_1) = \int_{\mathbb{R}^n} \frac{(v + \kappa b)(x_1) - (v + \kappa b)(y)}{|x_1 - y|^{n+2\sigma}} \, \mathrm{d}y$$

For any point $z \in B_1$ we know $(v + \kappa b)(x_1) \geq (v + \kappa b)(z)$. Let $A_0 = \{y \in B_1 \wedge v(y) \leq 0\}$. By assumption $|A_0| \geq \delta$. We use (4.2.5) and that $\kappa < 1/2$ to obtain the lower bound:

$$
\begin{aligned}
(-\triangle)^\sigma (v+\kappa b)(x_1) \geq & \int\limits_{y\in\mathbb{R}^n\setminus B_1} \frac{(v+\kappa b)(x_1)-(v+\kappa b)(y)}{|x_1-y|^{n+2\sigma}} \,\mathrm{d}y \\
& + \int\limits_{y\in B_1} \frac{(v+\kappa b)(x_1)-(v+\kappa b)(y)}{|x_1-y|^{n+2\sigma}} \,\mathrm{d}y \\
& \geq \int\limits_{\mathbb{R}^n\setminus B_1} \frac{1-|2y|^\alpha}{|x_1-y|^{n+2\sigma}} \,\mathrm{d}y + \int\limits_{A_0} \frac{1-\kappa}{|x_1-y|^{n+2\sigma}} \,\mathrm{d}y \\
& \geq \int\limits_{\mathbb{R}^n\setminus B_{1/4}} \frac{1-|4y|^\alpha}{|y|^{n+2\sigma}} \,\mathrm{d}y + \int\limits_{A_0} \frac{1/2}{|x_1-y|^{n+2\sigma}} \,\mathrm{d}y \\
& \geq \int\limits_{\mathbb{R}^n - B_{\frac{1}{4}}} \frac{1-|4y|^\alpha}{|y|^{n+2\sigma}} + \frac{\delta}{2\cdot 2^{n+2\sigma}}
\end{aligned}
$$

Therefore

$$
\varepsilon + \kappa \sup_x (-\triangle)^\sigma b(x) \geq \int\limits_{\mathbb{R}^n\setminus B_{1/4}} \frac{1-|4y|^\alpha}{|y|^{n+2\sigma}} \,\mathrm{d}y + \frac{\delta}{2\cdot 2^{n+2\sigma}}
$$

But this is a contradiction with (4.2.7). □

Remark 4.2.5. The proof of lemma 4.2.4 can be adapted to a more general family of operators instead of $(-\triangle)^\sigma$. In [13], it is used to obtain Hölder estimates for the corresponding equations. The operators for which the result applies include the case

$$
Tv(x) = \int\limits_{\mathbb{R}^n} a(x,y) \frac{v(x)-v(x+y)}{|y|^{\sigma(x)}} \,\mathrm{d}y
$$

for a bounded, symmetric $(a(x,y) = a(x,-y))$ and $0 < \sigma_0 \leq \sigma(x) \leq \sigma_1 < 1$. No modulus of continuity whatsoever is required either for a or σ. All the details can be found in [13].

Corollary 4.2.6. *For any $\sigma \in (0,1)$, if ε and α are chosen small enough then there is a $\gamma > 0$ such that if*

$$|(-\triangle)^\sigma v(x)| \leq \varepsilon \text{ for } x \in B_1$$
$$|v(x)| \leq 1 \text{ for } x \in B_1$$
$$|v(x)| \leq |2x|^\alpha \text{ for } x \in \mathbb{R}^n \setminus B_1$$

then $\operatorname{osc}_{B_{1/2}} v \leq 2 - \gamma$.

Proof. Consider the same γ as in lemma 4.2.4 for $\delta = \frac{|B_1|}{2}$. Suppose

$$|\{x \in B_1 : v(x) \leq 0\}| \geq \frac{|B_1|}{2},$$

otherwise we consider $-v$ instead of v. By lemma 4.2.4, we get $v(x) \leq 1 - \gamma$ for $x \in B_{1/2}$, we conclude $\operatorname{osc}_{B_{1/2}} v \leq 2 - \gamma$. □

Lemma 4.2.7. *For any $\sigma \in (0,1)$ and $\alpha \in (0, 2\sigma)$, if δ is close to $|B_1|$, then ε can be chosen small enough so that there is a $\gamma > 0$ such that if*

$$(-\triangle)^\sigma v(x) \leq \varepsilon \text{ for } x \in B_1$$
$$v(x) \leq 1 \text{ for } x \in B_1$$
$$v(x) \leq 1 + |2x|^\alpha \text{ for } x \in \mathbb{R}^n \setminus B_1$$
$$\delta \leq |\{x \in B_1 : v(x) \leq 0\}|$$

then $v(x) \leq 1 - \gamma$ for $x \in B_{1/2}$.

Proof. The proof is the same as in lemma 4.2.4 with the only difference that we have to choose κ such that:

$$\varepsilon + \kappa \sup_x (-\triangle)^\sigma b(x) + \int_{\mathbb{R}^n - B_{\frac{1}{4}}} \frac{|4y|^\alpha}{|y|^{n+2\sigma}} \, \mathrm{d}y < \inf_{\substack{A \subset B_1 \\ |A| = \delta}} \int_A \frac{1/2}{|x_1 - y|^{n+2\sigma}} \, \mathrm{d}y$$

for which we need δ close to $|B_1|$ so that the right hand side is larger than the last term of the left hand side. $\square$

Remark 4.2.8. If α, σ, δ and ε are a combination of constants for which Lemma 4.2.7 applies, then it also applies for lesser values of α. In other words, if it holds for one α, then it also holds for any α smaller.

Lemma 4.2.9. *For any $\sigma \in (0,1)$, let v be a function such that $(-\triangle)^\sigma v(x) = 0$ for x in some open set Ω. Suppose that*

$$|v(y) - v(x)| \leq c(|x-y|) \tag{4.2.8}$$

for every $x \in \mathbb{R}^n \setminus \Omega$, $y \in \mathbb{R}^n$ and some modulus of continuity c. Then the same holds for every $x, y \in \mathbb{R}^n$

Proof. We are left to show (4.2.8) when $x, y \in \Omega$. The function v is continuous in Ω because of the equation and in $\mathbb{R}^n \setminus \Omega$ because of (4.2.8), so v is continuous. Let $v_1(z) = v(z) - v(z + x - y)$, then $(-\triangle)^\sigma v_1(z) = 0$ for $z \in \Omega \cap (\Omega + y - x)$ and $v_1(x) \leq c(|x-y|)$ for $z \notin \Omega \cap (\Omega + y - x)$. By the maximum principle $v_1(z) \leq c(|x-y|)$ for every $z \in \mathbb{R}^n$, evaluating in $z = y$ we obtain the desired result. $\square$

4.3 Further regularity

Lemma 4.3.1. *Given $\mu > 0$ and u satisfying (4.1.6), if $u(x) \geq \mu r^2$ for one $x \in B_r$, then*

$$|\{x \in B_{2r} : u(x) > 0\}| \geq \delta\, |B_{2r}|$$

for some δ depending on μ.

Proof. We know that $u_{ee} \geq -C$ every time $|e| = 1$. In other words, u is semiconvex, for each point x there is a paraboloid touching u from below:

$$u(y) \geq u(x) + B \cdot (y - x) - \frac{C}{2} |x - y|^2$$

where B is any vector of the subdifferential of $u(y) + \frac{C}{2} |x - y|^2$ at x.

Now, let us consider the set:

$$A = \{y : B \cdot (y - x) \geq 0\} \cap B_{(\frac{\mu}{C})^{\frac{1}{2}} r}(x)$$

If $y \in B_{(\frac{\mu}{C})^{\frac{1}{2}} r}(x)$, then $\frac{C}{2} |x - y|^2 \leq \frac{\mu}{2} r^2$.

If $y \in A$, then

$$\begin{aligned} u(y) &\geq u(x) + B \cdot (y - x) - \frac{C}{2} |x - y|^2 \\ &\geq u(x) - \frac{\mu}{2} r^2 \\ &\geq \frac{\mu}{2} r^2 > 0 \end{aligned}$$

The set A is half of a ball. If $\frac{\mu}{C} \leq 1$, then A is going to be completely contained in B_{2r}, and we obtain the desired result with

$$\delta = \frac{1}{2} \left(\frac{\mu}{4C} \right)^{n/2}.$$

Otherwise we take $A' = \{y : B \cdot (y - x) \geq 0\} \cap B_r(x)$ instead of A and we obtain the desired result with

$$\delta = \frac{1}{2} \left(\frac{1}{2} \right)^{n}.$$

□

Remark 4.3.2. Lemma 4.3.1, as well as lemmas 4.2.4 and 4.2.7 can be applied to any two balls, one inside the other. The outer radius does not have to be double the inner radious. The lemmas, as stated, imply this by rescaling and a standard covering argument. And the proofs also clearly do not depend on the ratio between the radii. Of course the constants will vary.

We are now ready to start the proof of $w \in C^\alpha$.

Theorem 4.3.3. *Let u and w be like in (4.1.1-4.1.7), then w is C^α for a universal α, and its C^α norm depends on the various constants C of (4.1.4-4.1.7).*

Proof. Let us normalize u and w so that $\|w\|_{L^\infty} = 1$. We want to show that there is a constant $C_0 \geq 1$ such that for every $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{N}$,

$$\operatorname*{osc}_{B_{2^{-k}}(x_0)} w \leq C_0 2^{-\alpha k} \tag{4.3.1}$$

This clearly means that w is C^α.

We will show by induction that (4.3.1) holds for every k. The induction step works when $k \geq k_0$ for a large integer k_0. Then we can choose a large value for C_0 so that (4.3.1) holds for any $k \leq k_0$.

We can assume $x_0 = 0$. Let us also assume that $0 \in \operatorname{supp} u$, we will consider the case $x_0 \notin \operatorname{supp} u$ later. Suppose that (4.3.1) holds for $k = 0, 1, \dots, k_0$. Let us prove that it also holds for $k = k_0 + 1$. Let $\delta > 0$ be a small number to be determined later. We will prove that

$$\left| \left\{ x \in B_{2^{-k}} : w(x) - \inf_{B_{2^{-k_0}}} w \leq \frac{C_0}{2} 2^{-\alpha k_0} \right\} \right| \geq \delta \left| B_{2^{-k_0}} \right| \tag{4.3.2}$$

But first, let us show how (4.3.2) implies the inductive step.

Consider

$$v(x) = 2C_0^{-1}2^{k_0\alpha}\left(w(2^{-k_0}x) - \inf_{B_{2^{-k_0}}} w\right) - 1 \tag{4.3.3}$$

By (4.1.6), $-\triangle u \leq C$, thus $(-\triangle)^{1-s}v \leq C2^{k_0(\alpha+2s-2)}$. Then, if α was chosen smaller then $2-2s$, and k_0 is large enough, then v satisfies the hypothesis of lemma 4.2.4 with $\sigma = 1-s$. Therefore, there is a $\gamma > 0$ such that $v(x) \leq 1-\gamma$ for $x \in B_{1/2}$. Rescaling back to w we obtain $w(x) \leq C_0 2^{-k_0\alpha}(1-\frac{\gamma}{2}) + \inf_{B_{2^{-k_0}}} w$ for $x \in B_{2^{-k_0-1}}$. If α was chosen small enough so that $2^{-\alpha} \geq (1-\frac{\gamma}{2})$, then $\operatorname{osc}_{B_{2^{-k_0-1}}(x_0)} w \leq C_0 2^{-\alpha(k_0+1)}$. And the induction step is over.

We are left to prove (4.3.2). Suppose the contrary:

$$\left|\left\{x \in B_{2^{-k}} : w(x) - \inf_{B_{2^{-k_0}}} w \leq \frac{C_0}{2}2^{-\alpha k_0}\right\}\right| \leq \delta\,|B_{2^{-k_0}}| \tag{4.3.4}$$

We know $w(x) \geq \phi(x)$ for every x. Then $\inf_{B_{2^{-k_0}}} w \geq \inf_{B_{2^{-k_0}}} \phi$, and since ϕ is Lipschitz,

$$\operatorname*{osc}_{B_{2^{-k_0}}(x_0)} \phi \leq C2^{-k_0} < \frac{C_0}{2}2^{-\alpha k_0}$$

for k_0 large enough.

Every time $u(x) > 0$, $w(x) = \phi(x)$, therefore

$$\{x \in B_{2^{-k}} : u(x) > 0\} \subset \left\{x \in B_{2^{-k}} : w(x) - \inf_{B_{2^{-k_0}}} w \leq \frac{C_0}{2}2^{-\alpha k_0}\right\}$$

Thus

$$|\{x \in B_{2^{-k}} : u(x) > 0\}| \leq \delta\,|B_{2^{-k_0}}|$$

We choose δ in this proof to be small, so that the contrareciprocal of lemma 4.3.1 applies and we have $u(x) \leq \mu 2^{-2k_0}$ for every $x \in B_{\frac{3}{4}2^{-k_0}}$.

Let us consider the rescaled problem:

$$\bar{u}(x) = C_0^{-1} 2^{k_0(\alpha+2s)} u(2^{-k_0}x)$$
$$\bar{w}(x) = C_0^{-1} 2^{k_0\alpha} w(2^{-k_0}x)$$
$$\bar{\phi}(x) = C_0^{-1} 2^{k_0\alpha} \phi(2^{-k_0}x)$$

The pair $\bar{u}$ and $\bar{w}$ also satisfy the original hypothesis:

$$\bar{w}(x) = (-\triangle)^s \bar{u}(x)$$
$$\bar{u}(x) \geq 0$$
$$\bar{w}(x) \geq \bar{\phi}(x)$$
$$\bar{w}(x) = \bar{\phi}(x) \text{ when } \bar{u} > 0$$

From (4.1.4)-(4.1.7),

$$\sup_{x,y} \frac{|\bar{\phi}(x) - \bar{\phi}(y)|}{|x-y|} \leq C 2^{-k_0(1-\alpha)}$$
$$\bar{u}_{ee} \geq -C 2^{-k_0(2-2s-\alpha)} \text{ for every } e \text{ such that } |e| = 1$$

And also $\bar{u}(x) \leq C 2^{-k_0(2-2s-\alpha)}$ for every $x \in B_{3/4}$.

If k_0 is large enough, then

$$\sup_{x,y} \frac{|\bar{\phi}(x) - \bar{\phi}(y)|}{|x-y|} \leq \varepsilon \tag{4.3.5}$$
$$\bar{u}_{ee} \geq -\varepsilon \text{ for every } e \text{ such that } |e| = 1 \tag{4.3.6}$$
$$0 \leq \bar{u}(x) \leq \varepsilon \text{ for every } x \in B_{3/4} \tag{4.3.7}$$

for arbitrarily small ε. We choose ε much smaller than δ.

From (4.3.6) and (4.3.7), we conclude that u is Lipschitz in $B_{5/8}$ and its norm is less than $C\varepsilon$.

The inductive hypothesis, rescaled, turns into:

$$\operatorname*{osc}_{B_{2^k}} \bar{w} \leq 2^{k\alpha} \tag{4.3.8}$$

for $k = 0, 1, 2, \ldots$.

Then

$$|\bar{w}(x) - \bar{w}(0)| \leq |2x|^{\alpha} \qquad \text{for } |x| > 1 \tag{4.3.9}$$

$$|\bar{w}(x) - \bar{w}(0)| \leq 1 \qquad \text{for } |x| \leq 1 \tag{4.3.10}$$

We also know from (4.3.4) that

$$\left| \left\{ x \in B_1 : \bar{w}(x) - \inf_{B_1} \bar{w} > \frac{1}{2} \right\} \right| \geq (1 - \delta)\, |B_1|$$

Now, let b be a smooth cutoff function such that:

$$\begin{aligned} b(x) &= 0 && \text{for } x \in \mathbb{R}^n \setminus B_{5/8} \\ b(x) &\equiv 1 && \text{for } x \in B_{7/16} \\ b(x) &\leq 1 && \text{for every } x \in \mathbb{R}^n \end{aligned}$$

Thus

$$b(x)\, \bar{u}(x) \leq \varepsilon$$

$$(b\, \bar{u})_{ee} = b_{ee}\bar{u} + 2b_e\bar{u}_e + b\bar{u}_{ee} \geq -C\varepsilon$$

Let $h = (-\triangle)^s (b\,\bar{u})$. We can apply lemma 4.2.1 to obtain $h \le C\varepsilon$.

By construction $\bar{u} - b\bar{u} \equiv 0$ in $B_{7/16}$, therefore

$$0 = -\triangle(\bar{u} - b\bar{u}) = (-\triangle)^{1-s}(\bar{w} - h) \quad \text{ in } B_{7/16}$$

Let $v(x) = 1 + 2\left(h(x) + \inf_{B_1} \bar{w} - \bar{w}(x) - C\varepsilon\right)$. Then:

$$\begin{aligned}
(-\triangle)^{1-s} v = 0 & \qquad \text{in } B_{7/16} \\
\sup_{B_1} v \le 1 & \\
\sup_{B_{2^k}} v \le 1 + \left(2 \cdot 2^k\right)^{\alpha} & \qquad \text{for every positive integer } k \\
|\{x \in B_1 : v(x) < 0\}| \ge (1-\delta)\,|B_1| &
\end{aligned}$$

Then, if δ was chosen small, we can apply lemma 4.2.7 (rescaled) to v in the ball $B_{7/16}$ to obtain $v(x) \le (1-\gamma)$ for $x \in B_{1/2}$, that means

$$\bar{w}(x) \ge \gamma + \inf_{B_1} \bar{w} + h(x) - C\varepsilon \tag{4.3.11}$$

for x in $B_{1/2}$.

Let $v_1(x) = b(x) \cdot \bar{u}(x) + \varepsilon b(2x)$. Then $\max v_1(x) = v_1(x_0)$ for some $x_0 \in B_{1/2}$. Moreover, since $0 \in \operatorname{supp} \bar{u}$, and $\varepsilon b(2x)$ achieves constantly its maximum in a neighborhood of 0, then $\bar{u}(x_0) > 0$. Therefore $\bar{w}(x) = \bar{\phi}(x)$ for every x in a neighborhood of x_0, thus $\bar{w}$ and also $\bar{u}$ are sufficiently smooth at x_0 so that $(-\triangle)^s \bar{u}(x_0)$, as well as $(-\triangle)^s v_1(x_0)$, can be computed. Since x_0 is a maximum for v_1, $(-\triangle)^s v_1(x_0) \ge 0$. Therefore

$$\begin{aligned}
0 &\le (-\triangle)^s v_1(x_0) = h(x_0) + \varepsilon 2^s (-\triangle)^s b\,(2x_0) \\
&\le h(x_0) + C\varepsilon
\end{aligned}$$

Replacing in (4.3.11), we obtain:

$$\bar{w}(x_0) \geq \gamma + \inf_{B_1} \bar{w} - C\varepsilon$$

But since $x_0 \in \operatorname{supp} \bar{u}$, $\bar{w}(x_0) = \bar{\phi}(x_0)$, and $\inf_{B_1} \bar{w} \geq \inf_{B_1} \bar{\phi}$, then

$$\gamma \leq \bar{\phi}(x_0) - \inf_{B_1} \bar{\phi} - C\varepsilon \leq C' \varepsilon$$

But this is a contradiction if we chose the constants so that ε is much smaller than δ.

This finishes the proof for x in the support of u. To extend this modulus of continuity for all $x \in \mathbb{R}^n$ we observe that $(-\triangle)^{1-s} w = 0$ in the interior of $\{u = 0\}$ and we apply lemma 4.2.9 $\square$

4.4 The case when $\varphi \in C^{1,\beta}$

When we assume φ to be only $C^{1,\beta}$ in our obstacle problem, we can still obtain the result of Theorem 4.3.3 when we have $1 + \beta > 2s$. In order to show that, we have to improve some of the lemmas. We will explain the modifications in detail now.

Since we are not assuming $\varphi \in C^\infty$, we only have $\phi \in C^\tau$ for $\tau = 1 + \beta - 2s$. And instead of $u_{ee} \geq -C$, we have only a one sided $C^{1,\beta}$ estimate saying that for each $x \in \mathbb{R}^n$, there is a vector $a(x)$ such that

$$u(y) \geq u(x) + a(x) \cdot (y - x) - C\,|y - x|^{1+\beta} \tag{4.4.1}$$

Instead of Lemma 4.2.1 we will need the following lemma.

Lemma 4.4.1. *For $\sigma \in (0,\beta)$, there is a constant depending only on σ, β and dimension, such that if $\sup_x |v(x)| \leq A$ and for each x there is a $a(x)$ such that*

$$v(y) \geq v(x) + a(x)\cdot(y-x) - B\,|y-x|^{1+\beta} \quad \textit{for any } y \in \mathbb{R}^n$$

Then $(-\triangle)^\sigma v(x) \leq CB^{\frac{2\sigma}{1+\beta}} A^{\frac{1+\beta-2\sigma}{1+\beta}}$

Proof. The proof is very similar to the one of Lemma 4.2.1. We can also assume v to be smooth and do the computations.

$$\begin{aligned}
(-\triangle)^\sigma v(x) &= \int_{\mathbb{R}^n} \frac{v(x)-v(y)}{|x-y|^{n+2\sigma}}\,\mathrm{d}y \\
&\leq \int_{B_R(x)} \frac{v(x)-v(x)-\nabla v(x)\cdot(y-x)+B\,|y-x|^{1+\beta}}{|x-y|^{n+2\sigma}}\,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n\setminus B_R(x)} \frac{A}{|x-y|^{n+2\sigma}}\,\mathrm{d}y \\
&\leq B\left(\int_{B_R(x)} \frac{1}{|x-y|^{n+2\sigma-1-\beta}}\,\mathrm{d}y\right) + A\left(\int_{\mathbb{R}^n\setminus B_R(x)} \frac{1}{|x-y|^{n+2\sigma}}\,\mathrm{d}y\right) \\
&\leq C\left(B\cdot R^{1+\beta-2\sigma} + A\cdot R^{-2\sigma}\right)
\end{aligned}$$

Replacing R by $\left(\frac{A}{B}\right)^{\frac{1}{1+\beta}}$ in the above inequality we obtain:

$$(-\triangle)^\sigma v(x) \leq CB^{\frac{2\sigma}{1+\beta}} A^{\frac{1+\beta-2\sigma}{1+\beta}}$$

□

Instead of Lemma 4.3.1, we will use the following lemma that does not provide an estimate that is as good as before, but it is enough for our purposes.

Lemma 4.4.2. *Given $\mu > 0$ and u satisfying* (4.4.1)*, if $u(x) \geq \mu r^{1+\beta}$ for one $x \in B_r$, then*

$$|\{x \in B_{2r} : u(x) > 0\}| \geq \delta \, |B_{2r}|$$

for some δ depending on μ.

Proof. The proof is identical to the one for Lemma 4.3.1 but every time there is an estimate with a term $|x-y|^2$, $|x-y|^{1+\beta}$ has to be used instead. $\square$

A new lemma has to be added to replace the well known fact that semiconvex functions are locally Lipschitz.

Lemma 4.4.3. *Suppose*

$$|u(x)| \leq \varepsilon \qquad \text{for } x \in B_1 \tag{4.4.2}$$

$$u(y) \geq u(x) + a(x) \cdot (y-x) - \varepsilon \, |y-x|^{1+\beta} \qquad \text{for } x, y \in B_1 \tag{4.4.3}$$

Then u is Lipschitz in $B_{1/2}$ with a Lipschitz norm no greater than $C\varepsilon$, for a constant C.

Proof. For $x, y \in B_{1/2}$, let

$$K = \frac{|u(y) - u(x)|}{|x-y|}$$

Let us assume that $u(y) \geq u(x)$, so that $u(y) = u(x) + K\,|x-y|$. Let $z \in B_1$ be in the line determined by x and y so that $|x-z| \geq |y-z| \geq 1/2$. From (4.4.3), we have

$$u(y) - K\,|x-y| = u(x) \geq u(y) + a(y) \cdot (x-y) - \varepsilon\,|x-y|^{1+\beta} \tag{4.4.4}$$

$$u(z) \geq u(y) + a(y) \cdot (z-y) - \varepsilon\,|z-y|^{1+\beta} \tag{4.4.5}$$

From (4.4.4),

$$a(y) \cdot (y-x) \geq K\,|x-y| - \varepsilon\,|x-y|^{1+\beta}$$

From (4.4.5),

$$\begin{aligned}
u(z) &\geq -\varepsilon + a(y) \cdot (y-x)\frac{|z-x|}{|y-x|} - \varepsilon \\
&\geq a(y) \cdot (y-x)\frac{1}{2\,|y-x|} - 2\varepsilon \\
&\geq (K\,|x-y| - \varepsilon\,|x-y|^{1+\beta})\frac{1}{2\,|y-x|} - 2\varepsilon \\
&\geq \frac{K}{2} - 3\varepsilon
\end{aligned}$$

Since $u(z) \leq \varepsilon$ for any $z \in B_1$. Then $K < 8\varepsilon$. □

Now we are ready to state the theorem. Our assumptions now match what we know in the original obstacle problem when $\varphi \in C^{1,\beta}$.

Theorem 4.4.4. *Let u and ϕ satisfy* (4.1.1), (4.1.2) *and* (4.1.3). *Let $w = (-\triangle)^s u$. Let us also assume* (4.1.4) *and* (4.1.7), *and also*

$$C \geq \sup_{x,y} \frac{|\phi(x)-\phi(y)|}{|x-y|^{\tau}} \qquad \text{for } \tau = 1+\beta-2s \tag{4.4.6}$$

$$u(y) \geq u(x) + a(x)\cdot(y-x) - C\,|x-y|^{1+\beta} \tag{4.4.7}$$

$$(-\triangle)^{\sigma} u \leq C \qquad \text{for any } \sigma < \frac{1+\beta}{2} \tag{4.4.8}$$

where β is a positive real number such that $1+\beta > 2s$.

Then $w \in C^{\alpha}$ for an $\alpha > 0$ depending only on s, β and the dimension n.

Remark 4.4.5. In our original obstacle problem, we have (4.4.7) from Proposition 3.2.8 and (4.4.8) from Proposition 3.2.6. But actually (4.4.8) could be deduced from 4.4.7 and the boundedness of u by Lemma 4.4.1.

The proof is essentially the same as the proof of Theorem 4.3.3, with a few modifications.

First of all, for the function v defined in (4.3.3) we have to use Lemma 4.2.4 for a σ such that $0 < 2\sigma < 1 + \beta - 2s = \tau$. Since it is for those σ that we know $(-\triangle)^\sigma w = (-\triangle)^{\sigma+s} u \leq C$ and therefore $(-\triangle)^\sigma v \leq \varepsilon$ for k_0 large enough.

Later in the proof, when we construct the rescaled function $\bar{u}$, we will have a different upper bound for $\bar{u}$ in $B_{3/4}$ given by Lemma 4.4.2 instead of 4.3.1. We have $u(x) \leq \mu 2^{-(1+\beta)k_0}$ for $x \in B_{\frac{3}{4}2^{-k_0}}$, that is enough to obtain $0 \leq \bar{u}(x) \leq \varepsilon$ for $x \in B_{3/4}$ since $1 + \beta > 2s$.

Instead of (4.3.5) and (4.3.6), we have

$$\sup_{x,y} \frac{\left|\bar{\phi}(x) - \bar{\phi}(y)\right|}{|x-y|^\tau} \leq \varepsilon \qquad \text{for } \tau = 1 + \beta - 2s$$

$$\bar{u}(y) \geq \bar{u}(x) + \bar{a}(x) \cdot (y - x) - \varepsilon \left|y - x\right|^{1+\beta}$$

This is the point when we need to use Lemma 4.4.3 to obtain the Lipschitz bound for $\bar{u}$ in $B_{5/8}$ to be less than $C\varepsilon$.

Then the proof follows like in Theorem 4.3.3 until we have to estimate $(b\bar{u})_{ee}$ from below. Instead, we compute a one sided $C^{1,\alpha}$ estimate from the one of $\bar{u}$ and the smoothness of b. We have that

$$b(y) \geq b(x) + b'(x) \cdot (y - x) - C\,|x - y|^{1+\beta}$$
$$\bar{u}(y) \geq \bar{u}(x) + \bar{a}(x) \cdot (y - x) - \varepsilon\,|y - x|^{1+\beta}$$

Multiplying both inequalities and recalling that $\|\bar{u}\|_{Lip} \leq C\varepsilon$, we get

$$b(y)\bar{u}(y) \geq b(x)\bar{u}(x) + A(x) \cdot (y - x) - C\varepsilon\,|y - x|^{1+\beta}$$

where $A(x) = \bar{a}(x)b(x) + b'(x)\bar{u}(x)$.

Then we apply Lemma 4.4.1 instead of Lemma 4.2.1 to obtain $h = (-\triangle)^s(b\bar{u}) \leq C\varepsilon$.

The rest follows exactly as in the proof of Theorem 4.3.3.

Chapter 5

Towards optimal regularity

The observation that $(-\triangle)^{1-s}w = 0$ in the interior of the contact set $\{u = 0\}$ will allow us to estimate its growth in the free boundary by using a few barrier functions carefully. In this way, we will achieve optimal (or almost optimal) regularity results.

5.1 Barriers

In [9] a Poisson formula for the balayage problem of $(-\triangle)^\sigma$ is proven. The formula says that if g is a continuous function in $\mathbb{R}^n \setminus B_r$, then there exists a function u, continuous in $\mathbb{R}^n$, such that $u(x) = g(x)$ for every $|x| \geq r$, and $(-\triangle)^\sigma u(x) = 0$ for every $|x| < r$. The function u in B_r is given by the formula:

$$u(x) = \int\limits_{\mathbb{R}^n \setminus B_r} P(x,y) g(y) \, \mathrm{d}y$$

where

$$P(x,y) = C_{n,\alpha} \left(\frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^\sigma \frac{1}{|x-y|^n}$$

This is known as the balayage problem in B_r, and P is its corresponding Poisson function.

We can take $r \to \infty$ in the above formula to obtain a solution of the balayage problem in the semispace $\{x_n < 0\}$. If g is a continuous function in $\{x_n \geq 0\}$, then there is a function u, continuous in $\mathbb{R}^n$, such that $u(x) = g(x)$ for every x such that $x_n \geq 0$, and $(-\triangle)^\sigma u(x) = 0$ every time $x_n < 0$. The function u in $\{x_n < 0\}$ is given by the formula:

$$u(x) = \int_{\{y_n \geq 0\}} P(x,y) g(y) \, \mathrm{d}y$$

where

$$P(x,y) = C_{n,\alpha} \left(\frac{|x_n|}{|y_n|} \right)^\sigma \frac{1}{|x-y|^n}$$

The constant $C_{n,\alpha}$ is chosen so that

$$1 = \int_{\{y_n \geq 0\}} P(x,y) \, \mathrm{d}y$$

for any x (by rescaling, it is not hard so see that the above actually does not depend on x)

Notice that for each fixed y, P is C^σ across the boundary $\{x_n = 0\}$. We are going to construct barriers now to assure this regularity in several cases.

Let $g_0 = 1 - \chi_{B_1}$. Let $B(x) = g_0(x)$ when $x_n \geq 0$ and for $x_n < 0$ be given by the formula:

$$B(x) = \int_{\{y_n \geq 0\}} g_0(y) P(x,y) \, \mathrm{d}y = \int_{\{y_n \geq 0 \wedge |y| \geq 1\}} P(x,y) \, \mathrm{d}y \tag{5.1.1}$$

The function B would be the solution of the balayage problem in the semispace $\{x_n < 0\}$ for $B(x) = g(x)$ in $\{x_n \geq 0\}$ (See figure 5.1). In this case g is not continuous, so we just define B as the integral above.

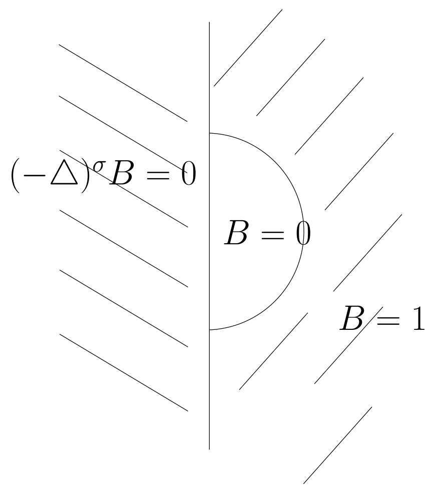

Figure 5.1: The function B.

Let us estimate now the behavior of B for small x. Let $|x| < 1/2$. Then if $|y| > 1$, $\frac{1}{2}|y| \leq |x-y| \leq 2|y|$, therefore

$$\begin{aligned} P(x,y) &= C_{n,\alpha}\left(\frac{|x_n|}{|y_n|}\right)^\sigma \frac{1}{|x-y|^n} \\ &\leq C_{n,\alpha}|x_n|^\sigma \frac{2^n}{|y_n|^\sigma |y|^n} \\ &\geq C_{n,\alpha}|x_n|^\sigma \frac{1}{2^n |y_n|^\sigma |y|^n} \end{aligned}$$

Applying these estimates to (5.1.1), we get

$$\frac{C}{2^n}|x_n|^\sigma \leq B(x) \leq 2^n C |x_n|^\sigma \tag{5.1.2}$$

for every $|x| < 1/2$, where C depends only on n and α and is given by

$$C = C_{n,\alpha} \int_{\{y_n \geq 0 \wedge |y| \geq 1\}} \frac{1}{|y_n|^\sigma |y|^n}\, \mathrm{d}y$$

On the other hand, when $|x| > 1$, it is clear that $1 \geq B(x) > d_1$, for some constant $d_1 > 0$ depending only on α and n.

Now, let $g_1(x) = \min(|x|, 1)$, let us solve the corresponding balayage problem for the semispace $\{x_n < 0\}$. We observe that $g_1(x) = \int_0^1 g_0(x/t)\, \mathrm{d}t$. Then

$$A(x) = \int\limits_{\{y_n \geq 0\}} g_1(y) P(x,y)\, \mathrm{d}y = \int\limits_0^1 B\left(\frac{x}{t}\right)\, \mathrm{d}t$$

We want to see the behavior of A when x approaches 0. Let $|x| < 1/4$,

$$\begin{aligned} A(x) &= \int\limits_0^1 B\left(\frac{x}{t}\right)\, \mathrm{d}t = \int\limits_0^{2|x|} B\left(\frac{x}{t}\right)\, \mathrm{d}t + \int\limits_{2|x|}^1 B\left(\frac{x}{t}\right)\, \mathrm{d}t \\ &\leq 2|x| + \int\limits_{2|x|}^1 2C\left(\frac{|x_n|}{t}\right)^\sigma\, \mathrm{d}t \\ &\leq 2|x| + \frac{2C}{r+1}|x_n|^\sigma (1 - (2|x|)^\sigma) \\ &\leq 2|x| + C|x_n|^\sigma \end{aligned}$$

On the other hand

$$\begin{aligned} A(x) &= \int\limits_0^1 B\left(\frac{x}{t}\right)\, \mathrm{d}t = \int\limits_0^{2|x|} B\left(\frac{x}{t}\right)\, \mathrm{d}t + \int\limits_{2|x|}^1 B\left(\frac{x}{t}\right)\, \mathrm{d}t \\ &\geq \int\limits_0^{2|x|} B\left(\frac{x}{t}\right)\, \mathrm{d}t \\ &\geq \int\limits_0^{2|x|} d_1\, \mathrm{d}t = 2d_1|x| \end{aligned}$$

The function A is continuous, and clearly $A(x) \to 1$ as $|x_n| \to \infty$. Let $\mu = \min_{\mathbb{R}^n \setminus B_{1/4}} A(x)$. Then

$$A(x) \geq \min(\mu, 2d_1|x|)$$

If we consider $\tilde{A}(x) = \frac{1}{\mu} A\left(\frac{\mu x}{2d_1}\right)$, then $\tilde{A}(x) \geq \min(1, |x|)$.

Proposition 5.1.1. *Let v be a continuous function in $\mathbb{R}^n$ such that*

1. *$(-\triangle)^\sigma v = 0$ in a convex open domain Ω.*
2. *$v(x)$ is Lipschitz and bounded in $\mathbb{R}^n \setminus \Omega$*

Then $v \in C^\sigma(\mathbb{R}^n)$

Proof. Without loss of generality, we can assume that the Lipschitz constant of v is 1 and $\|v\|_{L^\infty} = 1$. Let $x_0 \in \mathbb{R}^n \setminus \Omega$. Since Ω is convex, then there is a unitary matrix U such that that function $V(x) = \tilde{A}(U(x-x_0)) + v(x_0)$ satisfies $(-\triangle)^\sigma V = 0$ in Ω. Since $V(x) \geq \min(1, |x - x_0|) + v(x_0)$, then $V(x) \geq v(x)$ in $\mathbb{R}^n \setminus \Omega$. By maximum principle $V \geq v$ in the whole $\mathbb{R}^n$. In the same way we prove $v(x) \geq v(x_0) - \tilde{A}(U(x - x_0))$. Therefore we have a uniform C^σ modulus of continuity for every point x_0 in $R^n \setminus \Omega$. By lemma 4.2.9, $v \in C^\sigma(\mathbb{R}^n)$. □

5.2 Optimal regularity results

Optimal regularity can be quickly derived from what we have so far in the case when the contact set $\{u = 0\}$ is convex. The nonconvex case will require more work.

Theorem 5.2.1. *Let u like in Theorem 4.3.3, then if the interior of the contact set $\{x : u(x) = 0\}$ is convex, then $w \in C^{1-s}$, and therefore $u \in C^{1,s}$.*

Proof. By Theorem 4.3.3, $w \in C^\alpha$ for some small α. Then w is continuous is $\mathbb{R}^n$. Let Ω be the interior of $\{x : u(x) = 0\}$, that is convex. We also know

$w(x) = \phi(x)$ for every $x \in \mathbb{R}^n \setminus \Omega$, and ϕ is Lipschitz. Since $(-\triangle)^{1-s} w = 0$ in Ω, then we are in the conditions of proposition 5.1.1 with $\sigma = 1 - s$, so we conclude $w \in C^{1-s}$. $\square$

Remark 5.2.2. With a slightly different barrier function it could be shown that in the situation of Theorem 4.4.4, $w \in C^\alpha$ for $\alpha = \min(1-s, \tau)$ where $\phi \in C^\tau$.

Remark 5.2.3. By constructing test functions that solve the equation outside of a ball instead of a semispace, the above Theorem could be refined to a contact set that satisfies an exterior ball condition. For the time being we cannot assure any regularity for the free boundary. And such regularity theory is likely to require a sharp estimate on the regularity of the solution u.

The proofs from now on are not very different whether we consider ϕ to be Lipschitz or merely C^τ. We will thus describe the general case right away. We suppose that in our original obstacle problem $\varphi \in C^{1,\beta}$, and that $2s < 1 + \beta$, so that we have $\phi \in C^\tau$ for $\tau = 1 + \beta - 2s$.

The following lemma gives us an idea about how far from convex the level sets of u can be.

Lemma 5.2.4. *Let u be like in Theorem 4.4.4. Let's assume that $w \in C^\alpha$ for some given $\alpha < 1 + \beta - 2s$ (probably larger than the one from theorem 4.4.4). Let x_0 be a point such that $u(x_0) = 0$. Then for a small enough δ, there is a constant C_0 such that x_0 is not in the convex envelope of the set*

$$A_r = \left\{ x \in B_r(x_0) : w(x) > w(x_0) + C_0 r^{\alpha+\delta} \right\} \tag{5.2.1}$$

for any $r > 0$. Moreover δ can be chosen to be any positive real number less than $\left(\frac{1+\beta}{\alpha+2s} - 1\right)\alpha$ (for example one half of that).

Proof. Since $w \in C^\alpha$, then $u \in C^{\alpha+2s}$ (or $C^{1,\alpha+2s-1}$) so $w(x) = (-\triangle)^s u(x)$ can be computed by its integral representation.

We can assume $x_0 = 0$ without loss of generality.

We choose $\delta < \left(\frac{1+\beta}{\alpha+2s} - 1\right)\alpha$. Notice that $\delta < \tau - \alpha$ (recall $\tau = 1 + \beta - 2s$). If we take $C_0 = 1$ in the definition of A_r, we will prove that then the result of the lemma is true for r small enough. Choosing C_0 larger enough in (5.2.1), we can then make it true for larger values of r (Actually we can even make sure that A_r is empty for large r). So we will consider now

$$A_r = \left\{x \in B_r(x_0) : w(x) > w(x_0) + C_0 r^{\alpha+\delta}\right\} \tag{5.2.2}$$

and we want to show that for small enough r, 0 is not in the convex envelope of A_r.

Since ϕ is a C^τ function, $\phi(x) \le \phi(0) + C\,|x|^\tau$. Then, when $x \in A_r$ for r small enough

$$\phi(x) \le \phi(0) + C\,|x|^\tau \le w(0) + C_0 r^{\alpha+\delta} < w(x)$$

then $w(x) > \phi(x)$, thus $u(x) = 0$ for every $x \in A_r$.

Let us argue by contradiction. Suppose that we have k points $x_1 \dots x_k \in A_r$ such that a convex combination of them is 0.

$$\sum_{j=1}^{k} \lambda_j x_j = 0$$

for $\lambda_j \geq 0$ and $\sum \lambda_j = 1$.

For each j, we have

$$w(x_j) = \int\limits_{\mathbb{R}^n} \frac{u(x_j) - u(x_j + y)}{|y|^{n+2s}} \, \mathrm{d}y = \int\limits_{\mathbb{R}^n} \frac{-u(x_j + y)}{|y|^{n+2s}} \, \mathrm{d}y$$

Since u satisfies (4.4.7), at each point y there is a plane tangent from below with an error of order $C\,|z-y|^{1+\beta}$, i.e. there is a vector $A \in \mathbb{R}^n$ such that

$$u(z) \geq u(y) + A \cdot (z - y) - C\,|z - y|^{1+\beta} \tag{5.2.3}$$

for every $z \in \mathbb{R}^n$.

If we replace $z = x_j + y$ in (5.2.3) and add, we get

$$\sum_{j=1}^{k} \lambda_j u(x_j + y) \geq u(y) - Cr^{1+\beta} \tag{5.2.4}$$

Now we compute $w(0)$ and compare it with the values of $w(x_j)$

$$\begin{aligned} w(0) &= \int\limits_{\mathbb{R}^n} \frac{u(0) - u(y)}{|y|^{n+2s}} \, \mathrm{d}y = \int\limits_{\mathbb{R}^n} \frac{-u(y)}{|y|^{n+2s}} \, \mathrm{d}y \\ &= \int\limits_{B_{\tilde{r}}} \frac{-u(y)}{|y|^{n+2s}} \, \mathrm{d}y + \int\limits_{\mathbb{R}^n \setminus B_{\tilde{r}}} \frac{-u(y)}{|y|^{n+2s}} \, \mathrm{d}y \end{aligned}$$

For the first term we use that $w \in C^{\alpha}$, and then $u \in C^{\alpha+2s}$ (or $C^{1,\alpha+2s-1}$). Then $u(x) \leq C\,|x|^{\alpha+2s}$.

$$\int\limits_{B_{\tilde{r}}} \frac{-u(y)}{|y|^{n+2s}} \, \mathrm{d}y \geq -C\tilde{r}^{\alpha} \tag{5.2.5}$$

For the second term, we use (5.2.4)

$$\int_{\mathbb{R}^n\setminus B_{\tilde r}} \frac{-u(y)}{|y|^{n+2s}}\,\mathrm{d}y \geq \int_{\mathbb{R}^n\setminus B_{\tilde r}} \frac{-\sum_{j=1}^k \lambda_j u(x_j+y) - Cr^{1+\beta}}{|y|^{n+2s}}\,\mathrm{d}y$$
$$\geq \int_{\mathbb{R}^n\setminus B_{\tilde r}} \frac{-\sum_{j=1}^k \lambda_j u(x_j+y)}{|y|^{n+2s}}\,\mathrm{d}y - Cr^{1+\beta}\tilde{r}^{-2s}$$

Now we make the convenient choice for $\tilde r$. Let $\tilde r = r^p$, for $p = \frac{1+\beta}{\alpha+2s}$. We observe that since $p > \frac{\alpha+\delta}{\alpha}$, $x_j \in A_r$ and u is C^α, then for r small enough $u(x_j+y) = 0$ for every $y \in B_{\tilde r}$. Thus

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_{\tilde r}} \frac{-u(y)}{|y|^{n+2s}}\,\mathrm{d}y &\geq \int_{\mathbb{R}^n\setminus B_{\tilde r}} \frac{-\sum_{j=1}^k \lambda_j u(x_j+y)}{|y|^{n+2s}}\,\mathrm{d}y - Cr^{1+\beta}\tilde{r}^{-2s} \\ &\geq \sum_{j=1}^k \lambda_j w(x_j) - Cr^{1+\beta}\tilde{r}^{-2s} \\ &\geq w(0) + C_0 r^{\alpha+\delta} - Cr^{1+\beta-2sp}\end{aligned} \tag{5.2.6}$$

Adding (5.2.5) with (5.2.6), we get

$$w(0) \geq -C\tilde{r}^\alpha + w(0) + C_0 r^{\alpha+\delta} - Cr^{1+\beta-2sp} = w(0) + C_0 r^{\alpha+\delta} - Cr^{p\alpha} - Cr^{1+\beta-2sp}$$
$$\geq w(0) + C_0 r^{\alpha+\delta} - Cr^{\frac{(1+\beta)\alpha}{\alpha+2s}}$$

But this is impossible for small r because we chose δ so that $\alpha+\delta < \frac{2\alpha}{\alpha+2s}$. $\square$

Lemma 5.2.5. *Let u be like in Theorem 4.4.4. Let's assume that $w \in C^\alpha$ for some given $\alpha < \min(1-s, 1+\beta-2s)$ (probably larger than the one from theorem 4.4.4). Then w is actually in $C^{\gamma\alpha}$, where*

$$\begin{aligned}\gamma &= \frac{1-s}{1-s+\delta}\cdot\frac{\alpha+\delta}{\alpha} \\ &= \left(\frac{(1-s)(\alpha+2s+1+\beta)}{2(1-s)(\alpha+2s)+\alpha(1+\beta-\alpha-2s)}\right) > 1\end{aligned} \tag{5.2.7}$$

where δ is the one of lemma 5.2.4

Proof. First we will construct some auxiliary functions. Let B be as in (5.1.1) with $\sigma = 1 - s$. Recall that $B(x) \geq \beta$ when $|x| \geq 1$. Then, for $1 > r > 0$, let

$$D(x) = \frac{r^\alpha}{d_1} B\left(\frac{x}{r}\right) + \int_r^1 \frac{1}{d_1} B\left(\frac{x}{t}\right) t^{\alpha-1}\, \mathrm{d}t$$

Clearly, $D(x) = 0$ when $|x| < r$ and $x_n \geq 0$. When $|x| \geq r$,

$$D(x) \geq \frac{r^\alpha}{d_1} d_1 + \int_r^{\min(|x|,1)} \frac{1}{\alpha d_1} d_1 t^{\alpha-1}\, \mathrm{d}t \geq \min(|x|^\alpha, 1) \tag{5.2.8}$$

For $x_n < 0$ and $|x| < \frac{r}{2}$, applying (5.1.2),

$$\begin{aligned} D(x) &\leq C\left(r^\alpha \left(\frac{|x_n|}{r}\right)^{1-s} + \int_r^1 \left(\frac{|x_n|}{t}\right)^{1-s} t^{\alpha+1}\, \mathrm{d}t\right) \\ &\leq C\left(r^{\alpha+s-1} + 1\right) |x_n|^{1-s} \end{aligned} \tag{5.2.9}$$

Now, let us take a point in the free boundary $\partial\{u = 0\}$, that we will suppose to be the origin. Since w is C^α, for $0 < r < 1$, by (5.2.8) there is a constant C such that $w(x) - w(0) \leq CD(x)$ for every $|x| > r$. By lemma 5.2.4, if we choose r small enough, then $w(x) \leq w(0) + r^{\alpha+\delta}$ at least in half of the ball B_r. We can assume that $B_r \cap \{x_n \geq 0\}$ is that half of the ball. Therefore

$$w(x) \leq w(0) + r^{\alpha+\delta} + CD(x) \tag{5.2.10}$$

for every x except maybe some $x \in B_r \cap \{x_n < 0\}$.

Since ϕ is C^τ and r was chosen small, $\phi(x) \leq \phi(0) + C|x|^\tau = w(0) + C|x|^\tau \leq w(0) + r^{\alpha+\delta} + CD(x)$. Therefore all the points for which (5.2.10) does not hold must be in the set where $w(x) > \phi(x)$, i.e. in the interior of

$\{u(x) = 0\}$. But in that set $(-\triangle)^{1-s}w = 0$, also $(-\triangle)^{1-s}D = 0$ in that set (since it is included in $\{x_n < 0\} \cap B_r$. By maximum principle, (5.2.10) holds in the whole $\mathbb{R}^n$.

Let x be such that $|x|$ is small. Let $p = \frac{1-s}{1-s+\delta} < 1$, and $r = |x|^p > 2|x|$. Then, combining (5.2.10) with (5.2.9), we get

$$\begin{aligned} w(x) &\le w(0) + r^{\alpha+\delta} + C\left(r^{\alpha+s-1} + 1\right)|x_n|^{1-s} \\ &\le w(0) + |x|^{(\alpha+\delta)p} + C\left(|x|^{(\alpha+s-1)p} + 1\right)|x|^{1-s} \\ &\le w(0) + C\left(|x|^q + |x|^{1-s}\right) \le w(0) + C|x|^q \end{aligned} \tag{5.2.11}$$

where

$$q = (\alpha+\delta)p = \frac{(\alpha+\delta)(1-s)}{1-s+\delta} = \frac{(\alpha+s-1)(1-s)}{1-s+\delta} = (\alpha+s-1)p + (1-s)$$
$$< 1-s$$

Since $w(0) = \phi(0)$, then $w(x) \ge \phi(x) \ge \phi(0) - C|x| \ge \phi(0) - C|x|^q$ for $|x|$ small. And this C^q modulus of continuity holds at every point in the free boundary $\partial\{u = 0\}$.

Let x_0 be such that $u(x_0) > 0$, let x be any other point in $\mathbb{R}^n$. Let x_1 be a point in the segment between x and x_0 that is in the free boundary $\partial\{u = 0\}$. Then

$$\begin{aligned} |w(x_0) - w(0)| &\le |w(x_0) - w(x_1)| + |w(x_1) - w(x)| \\ &\le |\phi(x_0) - \phi(x_1)| + |w(x_1) - w(x)| \\ &\le C(|x_0 - x_1|^\tau + |x_1 - x|^q) \le C|x_0 - x|^q \end{aligned}$$

Thus, there is a uniform C^q modulus of continuity for every x in the set $\overline{\{u > 0\}}$. Since $(-\triangle)^{1-s}w = 0$ in the complement of this set, we can apply lemma 4.2.8 to conclude $w \in C^q(\mathbb{R}^n)$. Recalling that in lemma 5.2.4, δ could be chosen to be $\frac{1}{2}\left(\frac{1+\beta}{\alpha+2s} - 1\right)\alpha$, we get the complicated formula for q:

$$\begin{aligned} q &= \frac{(\alpha+\delta)(1-s)}{1-s+\delta} = \frac{1-s}{1-s+\delta} \cdot \frac{\alpha+\delta}{\alpha} \cdot \alpha \\ &= \left(\frac{(1-s)(\alpha+2s+1+\beta)}{2(1-s)(\alpha+2s)+\alpha(1+\beta-\alpha-2s)} \right) \alpha \\ &= \gamma\alpha \end{aligned}$$

□

Proposition 5.2.6. *Let u and w be like in Theorem 4.4.4. Then $w \in C^\alpha$ for every $\alpha < \min(1-s, 1+\beta-2s)$. Thus $u \in C^{1,\alpha}$ for every $\alpha < \min(s,\beta)$.*

Proof. From Theorem 4.3.3, we know than $w \in C^\alpha$ for some small $\alpha > 0$. Then we can apply lemma 5.2.5 repeatedly to get $w \in C^\alpha$ for larger values of α. To check that α gets as close to $\min(1-s, 1+\beta-2s)$ as desired we only have to observe that the application $I(\alpha) = \gamma\alpha$, where γ is given by (5.2.7) is continuous and such that $I(\alpha) > \alpha$ for every $\alpha \in (0, \min(1-s, 1+\beta-2s))$ and $I(\min(1-s, 1+\beta-2s)) = \min(1-s, 1+\beta-2s)$. □

Theorem 5.2.7. *Let $\beta > 0$. Given a function $\varphi \in C^{1,\beta}$, let u be the solution of the obstacle problem given by* (1.1.1)-(1.1.4). *Then $u \in C^{1,\alpha}$ for every positive number α less than* $\min(\beta, s)$.

Proof. In case $1+\beta > 2s$, we apply Proposition 5.2.6 to $u-\varphi$ with $\phi = -(-\triangle)^s\varphi$. Recall that $u-\varphi$ satisfies (4.4.7) and (4.4.8) because of Proposition 3.2.6 and Proposition 3.2.8, and (4.1.7) is satisfied because of Corolary 3.2.7.

In case $1+\beta \le 2s$, then the proof is simpler. From the definition of the problem (or from Proposition 3.1.1 if we start with the variational approach)

$(-\triangle)^s u \geq 0$ in $\mathbb{R}^n$. Therefore $(-\triangle)^\sigma u \geq 0$ in $\mathbb{R}^n$ for any $\sigma \leq s$, since $(-\triangle)^\sigma u = (-\triangle)^{\sigma-s}(-\triangle)^s u$ and $(-\triangle)^{\sigma-s}$ is given by the convolution with a positive kernel. Since $\varphi \in C^{1+\beta}$, $(-\triangle)^\sigma \varphi \in L^\infty$ for any $\sigma < \frac{1+\beta}{2}$, then from Proposition 3.2.6, $(-\triangle)^\sigma u \leq C$ in $\mathbb{R}^n$ for any $\sigma < \frac{1+\beta}{2}$. Thus $(-\triangle)^\sigma u \in L^\infty$ for any $\sigma < \frac{1+\beta}{2}$, and from Proposition 2.1.11 $u \in C^{1,\alpha}$ for any $\alpha < \beta$. $\square$

5.3 Some final remarks

As it was said in Remark 5.2.3, an optimal regularity result is achieved when the contact set has a uniform exterior ball condition. In case we had an interior ball condition, we could get the same result looking at the derivatives of u. If we take any direction e, we know that $\partial_e u$ is a C^α function for any $\alpha < s$. We also know that $(-\triangle)^s \partial_e u = 0$ in the set where $u > \varphi$, since ∂_e commutes with $(-\triangle)^s$. Then we could apply the barrier arguments to obtain $\partial_e u \in C^s$ as long as the contact set $\{u = \varphi\}$ has a uniform interior ball condition. We believe that the solution u to our problem is always $C^{1,s}$, although we have not been able to prove it in the general case yet.

In a very regular case, like for example a radially symmetric where the free boundary is a sphere, it is not hard to see that the regularity of the solution u does not get any better than $C^{1,s}$. Instead of using the barriers from above, we have to use them from below to show that $(-\triangle)^s u$ grows like C^{1-s} when it crosses the free boundary to the inside.

Our result can be extended to more general integral operators. The following Proposition gives a posible extension.

Proposition 5.3.1. *Let u be a continuous function solving the following obstacle problem:*

$$u \geq \varphi \qquad \text{in } \mathbb{R}^n \tag{5.3.1}$$

$$Lu \geq 0 \qquad \text{in } \mathbb{R}^n \tag{5.3.2}$$

$$Lu(x) = 0 \qquad \text{for those } x \text{ such that } u(x) > \varphi(x) \tag{5.3.3}$$

$$\lim_{|x| \to +\infty} u(x) = 0 \tag{5.3.4}$$

where φ is a smooth function with compact support and Lu is given by the integral formula

$$Lu(x) = \int_{\mathbb{R}^n} G(y)(u(x) - u(x+y))\,\mathrm{d}y$$

where G is a positive kernel such that $G(y) - \frac{1}{|y|^{n+2s}} \in L^1(\mathbb{R}^n)$. Then $u \in C^{1,\alpha}$ for every $\alpha < s$.

Remark 5.3.2. The condition $Lu \geq 0$ could be taken in the viscosity sense. These operators are related to jump processes in stochastic theory.

Proof. We notice that $Lu = (-\triangle)^s u + g * u$, where $g(y) = G(y) - \frac{1}{|y|^{n+2s}}$ is an L^1 function.

Since G is a positive kernel, it is a standard fact that the operator L satisfies the maximum (or comparison) principle, and it is also translation invariant. So all the results in chapter 3 apply to this case without change. Thus u is Lipschitz and semiconvex. Then $u * g$ is Lipschitz too. When we subtract $u - \varphi$ we arrive to exactly the same problem as in the beginning of chapter 4, where the right hand side is now $\phi = -(-\triangle)^s\varphi - g * \varphi - g * u$ that is a Lipschitz function. So the results of chapted 4 and 5 can be applied directly. $\square$

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Vita

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